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BERNOULLI, FERMAT AND THE BRACHISTOCHRONE AUGUST 27, 1991.

What follows are translations from the French, of two letters, one written by John Bernoulli and the other, a response to Pierre Fermat by a most hysterical Cartesian.

Bernoulli's is on the problem of quickest descent. The letter was discovered by C. Caratheodory who apparently publicized it on Aug. 31, 1936 at his presentation at the meeting of the Mathematical Association of America in Cambridge, Mass, during the tercentenary celebration of Harvard University.

In his presentation, Caratheodory states that, "the very first solution which John Bernoulli found for the problem of quickest descent contains the demonstration of the fact that the minimum is really attained for the cycloid." He adds that this method of Bernoulli's was ignored for nearly 200 years. As the letter to Pierre Fermat, from an irate Cartesian demonstrates, it was this evil Aristotelian faction that buried these discoveries for so long hoping they would never see the light of day again.

Bernoulli's letter was written in 1697 to a certain Mr. Basnage, who was the author of a history of the works of great thinkers.

Caratheodory writes that "after having told that the Philos. Transact. (No. 224, p. 384) for January 1697 contains a solution of the problem of quickest descent, which is presumably from Newton, Bernoulli writes:"

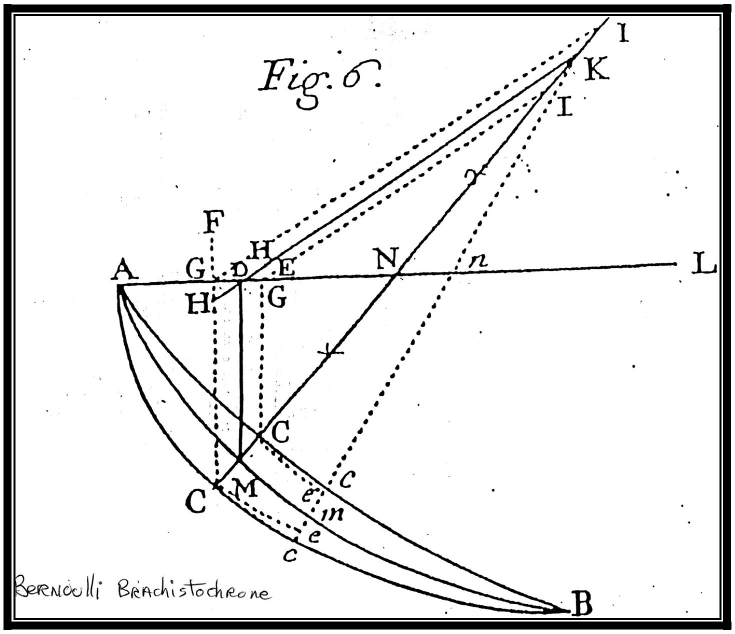
"I would only wish that Mr. Newton had done as we have, that is to say, that he had also published the method that had led him to the discovery of the sought after curve; because that is the way the public gains most: or, at least, if he had wanted to hide the analysis, he would not have done badly and would not do badly still, to confirm his construction by a synthetic demonstration, such as my method has furnished me; by which I prove demonstratively in the manner of the ancients, that there is only one curved line extending from one point to another, by which the heavy body descends in least time, and that this curve is the common (ordinary) cycloid, or as some have called it, the Roulette, which destroys entirely the ideas of a certain mathematician of high rank, who thought that there were many curved lines that could satisfy the above conditions.

"However, having found two different methods, one indirect and one direct, that deduces the resolution of the very foundation of the thing in considering the "maxima and minima", which led me to this synthetic demonstration; I have, however, published only the indirect; partly, because I believe it sufficient to convince those who would doubt the truth of these resolutions. Partly, also, because it gives, at the same time, the resolution of two famous problems of optics, that Mr. Huygens mentions in his "Treatise on Light", page 44, without daring to undertake the determination: to find the curvature of the ray of light, and the curvature of the wave of light, that is to say, the line that cuts perpendicularly all the rays projected from the same point of light. Because, I show, which is admirable, that if a transparent medium (diaphane) beginning by the point of light, and descending vertically changes continually in rarity, in proportion to the acquired speeds of a heavy body that falls from the same point of light, the curve of fastest descent will be precisely the same as that of the ray, that is to say, that both one and the other will be the Roulette or the cycloid; and the curve that I call "synchrone," and for which I give a very simple construction, namely, that which determines the portions, traversed in equal time, and on a same horizontal base, will be also perfectly the same as that of the wave, that is made in the aforesaid transparent medium (diaphane) by the radiating point: because both the one and the other will be perpendicular to their cycloids.

"It should also be noted that this identity of curves is not only found in the hypothesis of Galileo when the velocity of a falling body varies as the square root of the vertical heights; but, also, in any other hypothesis. So that, between these two speculations, taken from two so different parts of mathematics, as are Dioptrics and Mechanics, there exits an absolutely necessary and essential link.

"That is the reason why I have only shown to the light of day, the indirect way and suppressed still the direct; that is what Mr. Leibniz himself told me to do, seeing that the indirect way, as simple as it is, is of great consequence, and could nicely serve those who are accustomed to show off at the expense of others, as a means of making some little new discoveries, which should be sufficient for them to claim for themselves the possession and all of the glory of the invention. However, as I do not begrudge honest people anything, I communicated this method to Mr. Marquis de L'Hopital, who, like Mr. Leibniz, strongly approved of it, noting in it all kinds of surprising and extraordinary things. I, also, am not refusing to give it to whomever wishes it; one has only to write to me.

Figure 1.



Analytical Solution

"By the superior point A, from where a heavy body starts to fall to the other point B, draw the horizontal AL which is cut at N by any straight line INC forming any angle INL. From arbitrary point K on the part NI of the line INC, draw a line Knc so that the angle CKc formed is infinitely acute, such that the small arcs Ce, Mm, drawn from center K could be taken for small straight lines. All that I am going to do here shall be to find among an infinity of these small concentric arcs, which is the one that the heavy body, falling from A, can travel in the shortest time.

"For that, after having named NK, a; MN, x; and making the vertical MD be:

$$1 : m = MN\ (x) : MD\ (mx),$$

and

$$1 : n = CK : Ce = MK\ (x + a) : Mm\ (nx + na).$$

"Before going further, it is to be noted that there is nowhere there but the variable x; and that m, n are two numbers in which the first is finite and the second is infinitely small. That stated, we have

$$Mm : MD = (nx + na) : mx,$$

for the short segment of time by Mm, which should be here a little smaller, that is divided by the constant fraction n : m, and thereby different, should give accordingly

$$(x - a)dx : 2x \quad x = 0$$

from which results a = x. This is what shows that the nature of the curve AMB of the quickest descent, is to have, through whatever point M one wishes, the ray MK of its curvature, or of its osculating circle, cut in two equal parts by its axis AL: property that we know now for a long time does not match but the one cycloid. But when that is not already known, we can easily find it with our integral calculus.

"Following this method, the present problem can also be resolved more generally, namely, by supposing that heavy bodies, in falling, have their velocities not according to the square root of the vertical heights, as we ordinarily suppose, but rather, according to a function of those heights. If one calls mX this function of the height DM, and one takes it as above we have (x + a) : mX for a smaller for which the differential shall be consequently

$$(X - x\ X - a\ X)dx : XX = 0$$

...which gives

$$X = (x \quad a)\ X,$$

from which equation the root x will give the relationship of MN to NK; after which it will be left to integral calculus to reduce nature as is found in the curve, to an equation made of its coordinates: which is not to be done here.

Synthetic solution

"Given MK, mK, two perpendiculars to the cycloid AMB at two points M, m infinitely close to each other, these perpendiculars meet at point K of the evolute of this cycloid, and if extended these perpendiculars meet at C, c, any other curve ACB extended like the cycloid between two points A, B. After having imagined the little arc Ce described from center K, and then drawn MD, CG, perpendicular at D, G, to the horizontal AL; draw DK which if extended as well as CG (if it is necessary) cuts CG at H; and to which GI is parallel. And finally, on CG extended, draw CF third proportional to MD, CH.

That done, one has

$$MN = NK,$$

by the property of the cycloid we have similarly

$$CN = NI.$$

Then

$$CN + NK \quad 2CN \times NK,$$

and consequently

$$CN \; + NK \; + 2CN \times NK \;\; 4CN \times NK = CI \times MK.$$

Therefore having

$$CN + NK \; + 2CN \times NK = CK \, ,$$

we also have

$$CK \quad CI \times MK.$$

Which gives

$$MK : CK \;\; CK \; : CI.$$

Then

$$MK : CK = MD : CH \text{ (hyp)} = CH : CF.$$

And

$$CK : CI = CH : CG.$$

Therefore

$$CH : CF \quad CH : CG.$$

And consequently

$$CG \;\; CF.$$

Thus the small portion of time that the heavy body, fallen from the horizontal AL, requires to go the distance of the small arc Mm, is to the small portion of time that the same body falling from the same horizontal AL, would require to go the distance of the small arc Ce, in proportion to the straight lines of the small spaces Mm, Ce, and of the reciprocal of the square roots of the heights MD, CG,: that is to say, that the small portion of time through Mm, is to the small portion of

$$\text{time through Ce} = \frac{Mm}{MD} : \frac{Ce}{CG}$$

< and > because of the preceding hypothesis of MD, CH, CF, in continual proportion, gives

$$MD : CF = MD : CH = MK : CK = Mm : Ce$$

"the above value is equal to"

$$\frac{MD}{MD} : \frac{CF}{CG} = CG : CF$$

"Thus having found CG smaller than CF. we have here similarly the time through Mm shorter than the time through Ce, the hypotenuse of the triangle Cec. Thus, the time through all the elements Mm, that is to say, by the cycloid AMB, is shorter than the time through all the elements Ce, that is to say, than by any other curve ACB between the same points A, B, as is the cycloid AMB. Q.E.D."

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Caratheodory remarks that "the synthetic method of Bernoulli presents great analogies with the method of Weierstrass. If instead of using the normals to the cyloid he had made use of the curves he himself invented and called 'synchrones' the analogy would have become an identity."

Caratheodory recounts the following events regarding the battle between the Cartesians and Fermat:

In 1636 Descartes wrote his "Discours sur la methode de bien conduire sa raison." In Sept. 1637 Pierre Fermat "wrote at once to the Pere Mersenne who had presented him with the book, that he objected to the theory of Descartes because by this theory the velocity of light was supposed to increase with the density of the medium it was passing through. There ensued a long and tedious discussion, which lasted many years, but Fermat could not be persuaded, though the experiment showed that the law which Descartes had predicted for the refraction of light was accurate to the utmost."

In Aug. 1657, Fermat received a treatise about optics. Caratheodory writes that "In his letter in which he acknowledged the receipt of this book, he (Fermat) stated for the first time that the law of refraction might be deduced from a minimum principle".

In May 1662, Fermat received a letter from the Cartesian Clerselier, in which, says Caratheodory, "he was told that his principle, which is equivalent to what we call today the principle of least action, was at best a moral principle but not a principle of physics and that his theorem was simply a result of pure geometry."

Here is the letter:

From a letter from Clerselier to Fermat, May, 1662.

"Sir,

Do not think that I am answering you today because you think you have obtained the objective of troubling the peace of the Cartesians... Permit me just to tell you here the reasons that a zealous Cartesian could allege to preserve the honor and the right of his master, but not to give up his own advantage or to give you the initiative.

"1. The principle that you consider as the foundation of your demonstration, that is, that nature always acts along the shortest and simplest pathways, is nothing but a moral principle and not at all physical, that is, not and could not be the cause of any effect of nature.

"It is not, because it is not this principle that makes nature act, but rather, the secret force and the virtue that is in every thing, that is never determined by such or such an effect of this principle, but by the force that is in all causes that come together into one single action, and by the disposition that is actually found in all bodies upon which this force acts.

"And it could not be otherwise, or else, we would presume nature to have knowledge: and here, by nature, we mean only this order and this law established in the world as it is, which acts without foreknowledge, without choice and by a necessary determination.

"2. This same principle must put nature in an unresolved state, not knowing how to determine itself, when she has to pass a ray of light from a light medium through to a denser one. Because, I ask you, if it is true that nature must always act by the shortest and simplest pathways, since the straight line is undoubtedly both the shortest and the simplest of all, when a ray of light has to travel from a point of a light medium and end in a point of dense medium, isn't it the case that nature must hesitate, if you wish her to act by the princple of following a straight line soon after a break, since, if the latter is the shortest in time, the other is shorter and simpler in measure? Who will decide and who will pronounce himself on this matter?

"3. Since time is not what moves things it cannot either be that which determines movement, and once a body is moved and is determined to go in some direction, there is no apparent reason to believe that the time, more or less short, would force this body to change its determination, that which does not act and which has no power over it. But, since all speed and all determination of the movement of a body depends on the force and the disposition of that force, it is quite natural, and this is my belief, that it is better physics to say, as Mr. Descartes says, that the speed and determination of a body change because of the change occurring within the force and within the disposition of that force which are the real causes of its movement, and not to say, like you do, that they are changed by a design that nature has to always proceed by the pathway of least time, a design which she cannot have because she is unknowing and which cannot have any effect on the body."

Thus, Clerselier focuses the debate on the false assumption of the Cartesian "dualism": the separation of human reason (morality) from the physical world (nature).

This same objection to Bernoulli's and Fermat's method still continues today. In their textbook on mathematics, "What is Mathematics?" Richard Courant and Herbert Robbins write regarding Bernoulli: "Bernoulli’s 'proof' is a typical example of ingenious and valuable mathematical reasoning which, at the same time, is not at all rigorous. There are several tacit assumptions in the argument, and their justification would be more complicated and lengthy than the argument itself.... The question as to the intrinsic value of heuristic considerations of this type certainly deserves discussion, but would lead us too far astray."

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