A PEDAGOGICAL

TOWARDS UNDERSTANDING THE SCOPE OF PIERRE FERMAT'S "GREAT THEOREM" OF LEAST ACTION

by Pierre Beaudry

PART I

FERMAT'S "GREAT THEOREM"

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Lyndon LaRouche recently addressed the "great theorem" of Pierre Fermat by emphasizing that Fermat had bridged an axiomatic gap between the Archytas construction of the doubling of the cube and the Gauss "theorem of Algebra." LaRouche stated: "The indicated connection between the work of the ancient Pythagoreans, Plato, et al., and Gauss's 1977 doctoral dissertation, is found in the famous 'great theorem' of Pierre de Fermat, that it is impossible to determine so-called 'rational roots' for equations of greater than the second degree, a statement which is traced from the attempted treatment of cubic roots by modern mathematicians, such as the Sixteenth Century Cardano (Giralamo Cardano) et al. Hence the significance of the famous treatments on the subject of cubic and biquadratic functions by Carl Gauss." Lyndon H. LaRouche, Jr. {***What Connects the Dots?***}, January 21, 2006.

The following pedagogical exercise attempts to show how this "great theorem" challenge by Fermat is a unique expression of what LaRouche identified as Fermat's "definition of the 'quickest pathway' of refraction-reflection" for the universal physical principle of propagation of light. I do not intend to proceed as quickly here, but, relatively speaking, I will attempt to show how the "great theorem" opens the pathway of least action in understanding some of the more fundamental questions relative to the theory of numbers with respect to the said discovery of principle. There is no need to reinvent the wheel, in this matter, but there is a definite need to find a pathway to supersede it. In order to do that, it is useful to probe the "great theorem" in its original context. Consequently, I restate, here, literally, what Fermat had written, in Latin, as a marginal note, in his copy of Diophante's {***Arithmetica***}:

"It is impossible for a cube to be the sum of two cubes, or for a fourth power to be the sum of two other fourth powers, or, in general, for any power other than two to be the sum of two similar powers."

Fermat's idea was a direct response to the earlier challenge issued by Pythagoras, in confirming that the ancient Egyptians and Greeks had already understood that within the domain of {***Sphaerics***}, $A^2 + B^2 = C^2$ represented the only form of summation of powers.

In the same note, Fermat added:

"I have discovered a truly marvelous demonstration of this proposition, but this margin is too narrow to contain it."

Since that demonstration was never found, many empiricist ideologues, such as D'Alembert, Euler, and Lagrange, denied that Fermat ever had such a demonstration, and proceeded to obscure the issue, as LaRouche indicated, by pretending that Fermat was wrong, and that such cubic or higher roots could be found by way of "imaginary numbers;" a fraud that was disproved by Gauss in his doctoral dissertation of 1799. (Carl Friedrich Gauss, {***New Proof of the Theorem That Every Algebraic Rational Integral Function In One Variable can be Resolved into Real Factors of the First or the Second Degree***}, translated by Ernest Fandreyer, Helmstedt, At C.G. Fleckeisen's, 1799.)

The crucial point that I wish to make, here, is that Fermat's demonstration was never found because empiricists refused to look for it in Diophante, as LaRouche had earlier suggested people do. Indeed, the discovery of principle underlying the Fermat "great theorem" can be found in Diophantine arithmetic which implies that a cube could not be divided into two other cubes as a square can be decomposed into two other squares; a truth earlier geometrically proven by Archytas, in his construction of doubling of the cube, and later, algebraically demonstrated, by Gauss in his treatment of biquadratic rational roots of algebraic functions.

In his {***Arithmetica***}, Diophante proved the crucial point, in Book II, sections VIII, and IX, where Fermat had appended his note, and where Diophante showed his method of how to divide a square into two other squares. Diophante wrote:

"Divide a given square into two other squares. {***Example***} Let 16 be the given square, and I will call N^2 and $16 - N^2$ the sought for squares. N remains to be found, in such a way that $16 - N^2$ be a square. I pose that $16- N^2 = (2N – 4)^2$, thus $N = 16/5$." ({***Précis des oeuvres mathématiques de P. Fermat et de l'arithmétique de Diophante***}, par E. Brassinne, Imprimerie de Jean-Mathieu Douladoure, Toulouse, 1853, p. 53.) The interesting question, here is, how did Diophante derive 16/5?

This Diophantine problem can be resolved by applying the principle of proportionality that the American city surveyor of Washington D.C., Benjamin Banneker, had used for his mathematical puzzle. (See Pierre Beaudry, {***How Benjamin Banneker Discovered Proportionality in a Mathematical Puzzle. A Pedagogical***}, Leesburg, Va., October 39, 2003.) Banneker proved that such a principle, applied to both politics and astronomy, was also a seminal idea that he had used in building the plan for the Capital City. As the results showed, the Diophante method that Banneker had replicated was not a mere speculative exercise. Also, Leibniz had stressed the importance of Diophante as a source for his own infinitesimal calculus. As Leibniz wrote: "…I have therefore preferred to work for the common good, in the hope that others would spread the germs of my new theory and would reap brighter fruits than I had, especially if they were to apply themselves more seriously than has been done recently in developing Diophantine Algebra…" (G. W. Leibniz, {***Acta Eruditorum***}, May 1702.) For instance, it was from Diophante that Leibniz had devised his method of inversion of tangents, which he often used to discover new curves. While Diophante would say: "{***given a square, find in it two other squares***," Leibniz would say: "{***given the property of the tangent, find a curve.***}"

In a letter to Pascal written from Toulouse on August 29, 1654, Pierre Fermat showed a great interest in Pascal's treatise on the arithmetical triangle, and especially for his constructive proof of it. At the end of his letter, Fermat asked Pascal to use the Diophantine method and consider the following theorem of relating squares of the power of 2 to prime numbers:

> "The squares of the power of 2, increased by unity, are always primes.
> "The square of 2, increased by unity is 5, which is a prime.
> "The square of the square, 16, increased by unity, is 17, a prime.
> "The square of 16 is 256 which, increased by unity, is 257, a prime.

"The square of 256 is 65536 which, increased by unity, is 65537, a prime. And so on to infinity.

"I will answer to you for the truth of this property, but it is not easy to prove, and I confess that I have not yet been able to find a complete demonstration. I would not ask you to work at it if I had been successful."

I do not know if Fermat, or Pascal, ever found a constructive proof of this theorem, but it is definitely related to the "great theorem." Part of the answer to his query into the 256 series can be found by way of a self-reflexive inversion of the proposition, as per the Diophante method. That is, the squares of the powers of 2, diminished by unity, are divisible by all of the primes that are found through increasing such squares by unity, thus, making the power of 2 series a sort of complex arithmetic-geometric mean bridge between all prime numbers. So, the following can be added to the Fermat proposition.

The square of 2 is 4, decreased by unity, is 3, which is a prime.

The square of 4 is 16, decreased by unity, is 15, which is divisible by 5, and 3, both of which are primes.

The square of 16 is 256 which, decreased by unity, is 255, which is divisible by 17, 5, and 3, which are primes.

The square of 256 is 65536 which, decreased by unity, is 65535, which is divisible by, 257, 17, 5, and 3, all of which are primes. And so on to infinity.

In other words, though the discovering power must come before the primes are discovered, that discovering power cannot manifest itself without the discovery of the primes. It is like putting the cart before the horse. The reader should note that all great discoveries in history are made by means of such inversions, the most crucial of all, for solving today's strategic crisis being the discovery of principle of the 1648 Peace of Westphalia.

The reason there is such a unique closure by inversion, in both adding and decreasing the squares of the power of 2 by unity, resides in the fact that this series represents, among all possible series of rational whole numbers, the only one that provides perfect closure within self-bounding multiply connected circular action, and as a result, expresses all of the prime numbers and their multiples in the quadratic and biquadratic form of 4n + 1 and 2[4n + 1], as I will demonstrate next.

PART II:

THE GEOMETRIC DETERMINATION OF PRIME NUMBERS BY MEANS OF THEIR INTERVALS.

[First drafted on Wednesday, May 12, 2004]

The question of discovering the underlying geometry of prime numbers is as old as number theory itself, and one of the difficulties in discovering their generative principle is found in the epistemological flaw of treating numbers as things in and of themselves. As LaRouche often emphasized, music is between the notes. So, if we change the rules of the game and stop considering numbers pragmatically as if they represented loose change in our pockets, and begin to look at them as shadows of intervals of some form of productive physical action in the universe, we might be able to discover something fundamental about them.

Since there exists no possible geometry for things taken in themselves, as self-evident entities, the only way to solve that shortcoming is to eradicate the fallacy of considering numbers as things and start considering them as representing something else than what we have been brainwashed into thinking they are. A number is like money, it has no intrinsic value, and it is as stupid as money, in and of itself. Similarly, human beings are not a collection of stupid things that keep bumping into each other in the night or indulge in stupid competition with each other. Thus, from the vantage point of physical constructive geometry, the ordering principle of prime numbers should not be sought for in them, as such, but in the harmonic proportionality that lies between them. It is essentially the harmonic relationships between human beings that define economic science, just like it is the harmonic relationships of numbers that define the theory of numbers. It’s that simple.

As LaRouche has taught us, economics is the rate of increase of the productive powers of labor, which leads individual human beings into creating scientific and cultural objects that are necessary to improve the relative population density of our planet. Even though the world economy is more complex than prime numbers, their generative principle of reason and power is the same. So, let us apply the same universal physical principle of proportionality to the growth of prime numbers. By doing this, you will discover that it is the rate of increase of intervals between prime numbers that establishes their geometric ordering and the density of their distribution and growth, nothing else.

The following will demonstrate that the underlying, or rather the interlaying geometry of position of the intervals between prime numbers, forms patterns of well-ordered multiply-connected circular action, in which all of the multiples of the primes overlap each other as if in some braided form. This means that the patterns formed by the multiples of primes are but shadows of a process of physical action, which produces those braided patterns and which determines the distribution of the primes. We shall demonstrate that the principle of that rate of increase pertains to C-256.

ORDERING PRIME INTERVALS, AS SUCH.

First of all, order the prime numbers and their multiples according to a repeated simple series of intervals of 2 and 4. For instance, there is an interval of 2 between 5 and 7, and an interval of 4 between 7 and 11; then, again, there is an interval of 2 between 11 and 13, and an interval of 4 between 13 and 17, etc. This way, the multiples of all of the prime numbers will form well-ordered patterns within the lattice. Such patterns could be mapped on a plane, on a cylinder, on a cone, on a torus, or on a sphere, as if they were projected as a reflexion of a higher domain that produces them. However, they don't belong to the same domain that generates cubic roots, or higher roots. They appear to form circular patterns whose straight line shadows can be made visible on the plane of Figure 1, as if you were to "connect the dots" between all of them, and make them emerge from the dimly lit wall of Plato's cave.

Each series of prime multiples, say, for example, the multiples of 5, like 85, 125, 205, 245, 325, 365, 445, 485, etc., forms a unique pattern. Each prime has a unique pattern of multiples, and there exist as many patterns as there are prime numbers. All of them overlap each other and are interconnected by each other's intervals to form a coherent latticework. I let the reader illustrate this, for himself, by experimenting with Figure 1.

Figure 1. TABLE OF PRIME NUMBERS AND THEIR MULTIPLES.

5	7	11	13	17	19	23	25	29	31	35	37	41
43	47	49	53	55	59	61	65	67	71	73	77	7 9
83	85	89	91	95	97	101	103	107	109	113	115	119
121	125	127	131	133	137	139	143	145	149	151	155	157
161	163	167	169	173	175	179	181	185	187	191	193	197
199	203	205	209	211	215	217	221	223	227	229	233	235
239	241	245	247	251	253	257	259	263	265	269	271	275
277	281	283	287	289	293	295	299	301	305	307	311	313
317	319	323	325	329	331	335	337	341	343	347	349	353
355	359	361	365	367	371	373	377	379	383	385	389	391
395	397	401	403	407	409	413	415	419	421	425	427	431
433	437	439	443	445	449	451	455	457	461	463	467	469
473	475	479	481	485	487	491	493	497	499	503	505	509
511	515	517	521	523	527	529	533	535	539	541	545	547
551	553	557	559	563	565	569	571	575	577	581	583	587
589	593	595	599	601	605	607	611	613	617	619	623	625
629	631	635	637	641	643	647	649	653	655	659	661	665
667	671	673	677	679	683	685	689	691	695	697	701	703
707	709	713	715	719	721	725	727	731	733	737	739	743
745	749	751	755	757	761	763	767	769	773	775	779	781
785	787	791	793	797	799	803	805	809	811	815	817	821
823	827	829	833	835	839	841	845	847	851	853	857	859
863	865	869	871	875	877	881	883	887	889	893	895	899
901	905	907	911	913	917	919	923	925	929	931	935	937
941	943	947	949	953	955	959	961	965	967	971	973	977
979	983	985	989	991	995	997	1001					

Find the multiples of any prime number and "connect the dots" of their latticework. All of the multiples of each prime number can be generated in a similar fashion. For example, the multiples of 7 form simple diagonals across the lattice as they alternate by intervals of 8 and 4, such that between 7 and 35, for instance, there are 8 intervals, which are 11, 13, 17, 19, 23, 25, 29, 31, and between 35 and 49, there are 4 intervals, which are 37, 41, 43, and 47, etc. Again, the point to retain, here, is that it is not the values of those numbers in themselves that count, but the geometry of position of their intervals. So, what is the principle behind the construction of those patterns on intervals within the lattice?

THE DENSITY OF SINGULARITIES BETWEEN PRIMES

The density of intervals between the multiples of primes increases, as the prime numbers get larger. Note that the intervals between the multiples of 5 are 2 and 6, the intervals between the multiples of 7 are 4 and 8, and the intervals between the multiples of 11 are 6 and 14, and so forth. The intervals of those intervals grow arithmetically by even numbers as all of them increase consistently by 4. Thus, we can establish a table of the different intervals of the multiples of primes, including the quadratic intervals between those intervals. [Figure. 2]

Figure 2

PRIMES & MULTIPLES			**INTERVALS**		**INTERVALS OF INTERVALS**	
5			[2 and 6]		4	
	2	2		2		
7			[4 and 8]		4	
	4	2		6		4
11			[6 and 14]		8	
	2	2		2		
13			[8 and 16]		8	
	4	2		6		
17			[10 and 22]		12	
	2	2		2		
19			[12 and 24]		12	
	4	2		6		4
23			[14 and 30]		16	
	2	2		2		
25			[16 and 32]		16	
	4	2		6		
29			[18 and 38]		20	
	2	2		2		
31			[20 and 40]		20	
	4	2		6		4
35			[22 and 46]		24	
	2	2		2		
37			[24 and 48]		24	

	4	2		6			
41			[26 and 54]			28	
	2	2		2			
43			[28 and 56]			28	
	4	2		6			4
47			[30 and 62]			32	
	2	2		2			
49			[32 and 64]		Etc.		

As the value of a prime number increases, so do, proportionately, the intervals of their multiples, up to a certain limit. The first series of intervals of multiples increases by alternating factors of 2 and 6, the second series of intervals of intervals grows by a constant factor of 4. This second growth rate, that is, the intervals of intervals, represents a second derivative form of action, following an underlying ordering of quadratics, that is, an ordering of the second degree, from within which the musical octaves, based on C-256, is derived. This second degree of intervals changes, because the ratio of intervals increases by a conical spiral function, which distributes the primes in accordance with the power of 2.

This does not mean that things stop growing at that point, but that the system has reached a limit, a boundary condition dominated by second degree quadratics, whose heads pierce through the cracks of the universe to show that they belong to some infinite and universal object which eludes the grasp of our sense perception. It is this constant ratio of the power of 2 within quadratics, this invariant interval of interval factor of doubling, which determines the density of distribution of prime numbers.

PART III

A REFLEXION ON THE APRIL 27, 2002 PEDAGOGICAL OF JONATHAN TENNENBAUM CONCERNING ARTICLES 107- 108 of {*DISQUISITIONES ARITHMETICAE*} BY CARL FRIEDRICH GAUSS.

[First drafted in Leesburg, May 8, 2002.]

As Jonathan demonstrated in his pedagogical, Gauss had challenged the scientific community to solve an elementary, but difficult problem of discovering the ordering principle for the distribution of prime numbers, given only a single shadow projected on the dimly lit wall of Plato's cave. The discovery of this ordering principle had led Gauss to develop the fundamental theorem of Algebra, which gave a devastating refutation of D'Alembert, Leonard Euler and Jean-Louis Lagrange's method of algebraic empiricism. This is the way Gauss formulated his crucial concept, again, using the method of Diophante:

>107. "It is very easy, given a modulus, to characterize all the numbers that are residues or nonresidues…But the inverse question, {***given a number, to assign all numbers of which it is a residue or a nonresidue,***} is much more difficult." To wit:

>108. "-1 is a quadratic residue of all numbers of the form 4n+1 and a nonresidue of all the numbers of the form 4n+3." (Carl Gauss, {***Disquisitiones Arithmeticae***}, p. 71-72)

Think of this question, as a sort of "boot-strap" principle of self-generation, in which the process contains the paradox of Diophante; that is, if the effectiveness of this inversion were not true, there would be an absolute limit beyond which one could not go. However, one is able to go beyond and contradict the limit to show there exist higher-boundedness of action in the universe. So, in that unique way, the universe is "limited and yet increasingly self-bounded." In other words, Gauss is showing how to solve the Leibniz anti-empiricist proposition: {***given the property of a tangent, find the curve.***}

To discover this, Jonathan proposed to proceed in the following manner: Since subtracting -1 from a number has the same significance as adding 1 to it, take the series of squares: 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, etc., which represents the entire series of rational whole numbers, and add 1 to each number such that they are now transformed as:

5, 10, 17, 26, 37, 50, 65, 82, 101, 122, etc.

When you look at this last series, you realize that all primes of the 4n+1 species form half of that series of squares +1, that is, 5, 17, 37, 65, and 101, but such is not the case for the other half of those numbers. My surprise was to find that those other numbers were not directly derivable by means of the form [4n+3] either. I was perplexed, until I found that the other numbers of that series, 10, 26, 50, 82, and 122, were all of the form of a close relative of 4n+1, which is 2[4n+1], a biquadratic. Even though Gauss had pursued the matter by demonstrating that –1 was a residue to a prime modulus of the biquadratic form of 8N + 1, I did not see how helpful this would be until I found the ordering principle behind the numbers of the biquadratic form of the square root of 2[4n+1]-1. That looked to me as a very curious sort of complex number.

As the history of science shows, a discovery of physical principle is, most of the time, caused by a nagging anomaly, a real ambiguity that keeps knocking in the back of your mind. My investigation into the series of numbers in the form of the square root of 2[4n+1]-1 led to the realization that these numbers could integrate the two species of primes, [4n+1] and [4n+3], that Gauss was talking about. So, I asked myself: What if the principle underlying this form of the square root of 2[4n+1]-1, alone, were to provide a least action pathway to integrate all of the primes dressed up in the form of squares, in accordance with Fermat, Diophante, and Pythagoras? I first lined up the series of numbers as follows to look at the ordering principle of the change between them, in my mind's eye:

5	=	[4x1+1]
10	=	2 [4x1+1]
17	=	[4x4+1]
26	=	2 [4x3+1]
37	=	[4x9+1]
50	=	2 [4x6+1]
65	=	[4x16+1]
82	=	2 [4x10+1]
101	=	[4x25+1]
122	=	2 [4x15+1] etc.

Figure 3.

I noticed that all of the values of [4n+1] corresponded to squares +1, that is, [4x1+1], [4x4+1], [4x9+1], [4x16+1], [4x25+1], forming directly the series of numbers 5, 17, 37, 65, 101. However, if one pays attention to the other numbers of the form 2[4n+1], that is, a process of simple doubling of primes, as opposed to a squaring, a most fascinating series of intervals emerges, and a series of intervals of intervals takes hold of the whole process, whose harmonic ordering generates all of the primes!

The intervals between each multiple within the 2[4n+1] brackets, that is, between 1, 3, 6, 10, 15, have intervals of 2, 3, 4, 5, an increase by 1, while the intervals of intervals between the corresponding numbers 10, 26, 50, 82, 122, are 16, 24, 32, 40, thus expressing a constant increase by 8, a biquadratic interval. I will show that there exists, here, a {***harmonic ordering principle of these complex biquadratic intervals***} which generates all of the primes from any of the modules of the power of two, that is, the 256 series.

NUMBER THEORY AND TORUS GEOMETRY

From this vantage point, it became clear to me that the {***intention of numbers***}, established by God himself, so to speak, had never been to empirically count things as self-evident things, as if you were counting your loose change, but to express a different kind of change, a change in direction, a change of orientation, even a change of axioms by means of intervals of physical action; that is, by means of adding a new dimensionality to the cycles of universal physical processes. In this case, the added dimensionality is that of the physical geometry of the torus. Thus, what was implied here, was a physical principle of action such as that of a least action form of the catenoid curvature of a torus which functions in such a way that it is motionless, yet is, paradoxically, everywhere in motion.

For example, take any module of the 256 series, say 16, and put it to work into a complex circular form of a torus. It doesn't move, but it changes all the time.

1[3]   11[23]
*   *   *
0   2[5]
*   →   *
5[11]   8[17]
*   *
12[25]   6[13]
*   ↑   16   ↓   *
9[19]   3[7]
*   *
7[15]   10[21]
*   ←   *
13[27] *   *   * 15[31]
4[9]   14[29]

Figure 4

First, mark the simple 16 intervals of action by whole numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, clockwise, count every unit of interval of action starting from 0 to 1, noting that you are increasing each interval of action by one, after each number, as you move forward, clockwise. Fill the entire module until all of the 16 intervals are covered.

Next, overlap this first series of simple intervals by a second series of the complex numbers of the form square roots of 2[4n+1]-1, which changes the direction, ever so slightly, by adding a new dimensionality. Go from the square root of 2[4x1+1]-1 up to the square root of 2[4x120+1]-1. This generates all of the primes from a second-degree derivative interval of intervals, which overlaps the intervals of the first degree, binds them together proportionately, and modifies their course by an infinitesimal degree of change in each interval of action. This should demonstrate the point made by Fermat, which is that, with this least action process of the torus, you have reached the boundary condition of all rational numbers with roots of the second degree. Thus, as if in a fast spinning sun, the self-bounding physical geometry of the process of change is reflected in the characteristic of the torus.

For example, the first prime in the series is the square root of 2[4x1+1]-1, which is [3], because the square root of 10-1 is [3]. That is your first prime, whose value overlaps the first interval from 0 to 1 in the circle of Figure 5, thus, modifying its position course ever so slightly.

The second prime in the series is the square root of 2[4x3+1]-1, which is [5], because the square root of 26 - 1 is [5]. That is your second prime, whose value overlaps the two intervals between 1 and 2 in the circular form, and maintains the same changed course.

The third prime in the series is the square root of 2[4x6+1]-1, which is [7], because the square root of 50-1 is [7]. That is your third prime, whose value overlaps the three intervals between 2 and 3 in the circular form, and maintains the same changed course.

The missing biquadratic square roots of 2[4x2+1]-1, of 2[4x4+1]-1, and of 2[4x5+1]-1, are not rational whole numbers, and therefore, have simply been overshadowed, leaving the necessary spaces open for subsequent primes to come later and fill their pre-established harmonic positions within the cycle of the modular function.

The remaining primes (and multiples of primes) found in this manner shall be, [9], [11], [13], [15], [17], [19], [21], [23], [25], [27], [29], [31], all of which shall find their pre-assigned intervals in due course within this cycle. Although there exists a simple formula to achieve such results, it is better to construct these biquadratic second-degree {***intervals of intervals***} long hand, and enjoy the complex beauty of their relationships in the geometry of your mind. This way, you experience the amount of playful least action work that God has put into them.

In a nutshell, this exercise intended to show how the "great theorem" of Fermat implicitly led us to generate and distribute all of the prime numbers in a Diophantine biquadratic form of the square root of 2[4n+1]-1, and thus, enabling us to develop all rational roots of the second degree as a {***torus modular function***}. As LaRouche indicated in {***What Connects the Dots***}, the "great theorem" of Fermat is not merely mathematical, as such, but "ontological" in character, that is, it pertains to the universal principle of least action underlying all physical processes, including primarily economic processes. This is why Fermat was right in establishing a limitation whereby it is impossible to divide any power into two other powers of the same order beyond the second degree, because the universe is "finite and yet not bounded." pierrebeaudry@larouchepub.com

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