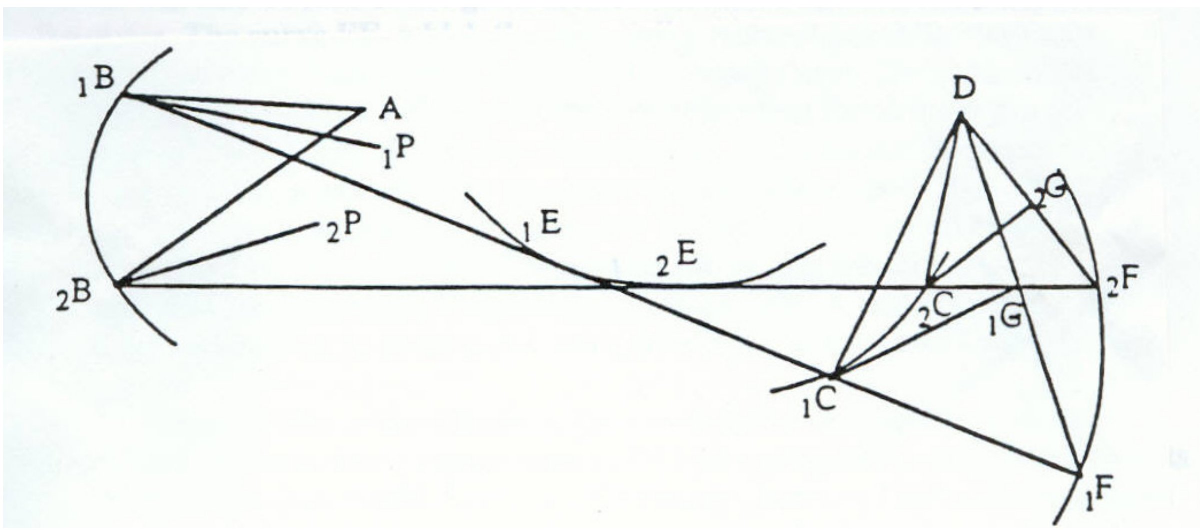
THE DISCOVERIES OF PRINCIPLE OF THE CALCULUS
IN
(ACTA ERUDITORUM)

G.W. Leibniz



8 EXTRACTS translated by Pierre Beaudry
LEESBURG, VA, U.S.A. SEPT. 2000

COVER ILLUSTRATION: LEAST ACTION PRINCIPLE. OF LIGHT PROPAGATION

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G.W. LEIBNIZ

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THE SPIRIT OF INVENTION AND THE LEAST ACTION PRINCIPLE

by Pierre Beaudry

> ***"Nature is just; she equally distributes all that is necessary to the individual, put on earth to live, work and die; she reserves to a small number of human beings, however, the right to enlighten the world, and by entrusting them with the lights that they must diffuse across their century, she says to one, you shall observe my phenomena, to the other, you shall be a geometer: she calls on this one for the purpose of legislation; she calls on this other one to paint the morals of people, of revolutions, and of empires. These geniuses pass away after they have perfected human reason, and leave behind them a great memory. But all of them have traveled on different routes: only one man elevated himself, and dared to become universal, a man whose strong will synthesized the spirit of invention, and the spirit of method, and who seemed to have been born to tell the human race: behold, and know the dignity of your species! These are the traits by which Europe has given recognition to Leibniz..." [Jean Sylvain Bailly, ELOGE DE LEIBNIZ, 1779]***

After more than 300 years of deliberate and systematic suppression, by the scientific community, of the* {ACTA ERUDITORUM} *writings of Gottfried Leibniz, it is an honor and a privilege to have the opportunity of publishing, for the first time in the English language, a few papers of one of the greatest thinkers of all times. For three centuries, only a handful of people have had the opportunity to access this extraordinary source of knowledge and invention, and therefore, the youth of today is very fortunate to have access to these rich and provoking ideas of one of the greatest minds that ever lived.

The purpose for publishing this series of articles on the calculus is not to have the student master a new mathematical tool, or have him learn formulas for manipulating the calculus, but rather to have him discover the underlying geometry of developing the power of invention, and of expanding the powers of his mind. From this vantagepoint, the study of Leibniz's* {TRANSCENDENTAL GEOMETRY}*, known otherwise as his* {CALCULUS}*, is aimed at discovering a* {STRATEGY OF FLANKING ACTION}*, which increases the* {RELATIVE POTENTIAL POPULATION DENSITY} *of humanity, and raises humanity to a higher level of power over the universe. Such was the intention of Leibniz in developing the* {LEAST ACTION PRINCIPLE}*. The crucial discovery of* {FLANKING LEAST ACTION}*, developed by Leibniz in* {ACTA ERUDITORUM}*, as opposed to* {ACTION AT A DISTANCE} *concocted by Galileo, and Newton, will show that the fundamental issue is not to find the "equation of a curve", or even the "quantity of curvature of a given curve," but to determine the* {LEAST

ACTION MEASURE OF CHANGE IN CURVATURE} ***of physical space-time. As Lyndon LaRouche identified the problem for us, more than ten years ago:***

"{THE PROBLEM OF INTEGRATION OF LEAST ACTION TO MEASURE THE WORK ACCOMPLISHED BY HIGHER ORDER LEAST ACTION, IS DERIVED FROM THE FALLACY OF SEEKING A SCALAR MEASUREMENT OF THE WORK SO ACCOMPLISHED, AND THAT RELATED FALLACY OF ATTEMPTING TO MEASURE THE DISPLACEMENT ASSOCIATED WITH THE ACTION IN A SCALAR OR ANALOGOUS WAY. WHAT WE MUST MEASURE ARE:

1- "INCREASE OF DENSITY OF DISCONTINUITIES PER INTERVAL OF ACTION, RELATIVE TO THE DENSITY OF SINGULARITIES IN THE DOMAIN INTO WHICH THE NEW LEAST ACTION IS INTRODUCED. THIS IS ANALOGOUS TO ADDING NEW DEGREES OF FREEDOM IN THE MANNER SUGGESTED BY THE RIEMANN SURFACE FUNCTION.

2- "THE RESULT OF THE ACTION MUST BE MEASURED AS A CHANGE IN THE CURVATURE OF PHYSICAL SPACE-TIME.}"[LYNDON H. LAROUCHE JR. {INTERNAL MEMO}, SEPT. 7, 1988.]

Thus, the question becomes an existential-political question, a life and death question in the most profound sense for the survival of civilization, because the way you **{MEASURE CHANGE IN THE CURVATURE OF PHYSICAL SPACE-TIME},** ***especially in a severe period of economic and social crisis like today, will determine the sort of world your children, and your grandchildren, yet unborn, will have to live in: either automatons in an oligarchical one world government, or free cognitive individuals in sovereign republican nation-states.*** **{OLIGARCHICAL PSEUDO-SCIENCE}** ***represented by Aristotle, Newton, Descartes, Euler, Maupertuit, LaPlace, Cauchy, Von Newman et al, is a proven failure, because it always attempted to measure curvature linearly, and objectively, by scalar quantities, and in adopting such*** **{BENCHMARKING},** ***denied the creative power of the human mind. These so-called scientists were unable to measure a universe of change.*** **{REPUBLICAN SCIENCE},** ***on the other hand, represented by Plato, Cusa, Kepler, Leibniz, Carnot, Monge, Kaestner, Gauss, Riemann, and LaRouche, is grounded on the fundamental axiom that the universe is knowable, changes all the time, and that, therefore, man's creative powers are always challenged to discover a new and higher*** **{MEASURE OF CHANGE},** ***a change measured by non-linearity, and*** **{COGNITIVE SUBJECTIVITY}.** ***The question then becomes: which approach will you chose?***

At the onset of any great discovery, be it the discovery of the solar hypothesis, the discovery of the well tempering of the musical system, or the discovery of the least action curvature of light propagation, there generally appear crucial anomalies, and devastating paradoxes, that baffle and shock the scientific world, because they cannot be explained by the old rules, and therefore, they have the effect of eroding and undermining the fixed underlying axiomatic beliefs of the popular opinions that control the previous accepted world view. The discovery of the Leibnizian calculus has had such an effect, and has challenged not only the

underlying axiomatic beliefs of his own time, but also of the entire scientific community for the last three hundred years to this day.

As Lyndon LaRouche has been emphasizing for several decades about the epistemological nature of the problem of mathematics in general, but of algebra in particular, it was Leibniz and Jean Bernoulli who established that physics had to be grounded on a non-algebraic mathematics, a **{TRANSCENDENTAL GEOMETRY},** *and especially from the standpoint of Cusa and his revolutionary discovery of the solution to the ontological paradox of squaring the circle. The notion of physical non-linear-least-action principle as opposed to action at a distance is firmly established in the Acta Eruditorum series that Leibniz published during a period of about 30 years, from 1682 to 1713. The Leibniz calculus, however, did not appear all of a sudden, emerging already formed, as if from the head of Jupiter. What emerged with Gilles de Roberval, Blaise Pascal, Pierre de Fermat, and Christian Huygens, represented the initial steps of such a revolution in the domain of* **{TRANSCENDENTAL GEOMETRY},** *a geometry of optimism, as oppose to the pessimism of Descartes and Newton. This revolution represented a voluntarist policy of thinking big in opposition to the fatalistic small thinking of Descartes et al. Thus, Leibniz restored the Rabelaisian truth of enduring time whereby* **{"HISTORY LEADS THE WILLING, THE UNWILLING DRAGS!"}**

In the early 1660's, each in his own way, the members of the newly created French Academy of Sciences, Huygens, Fermat, Roberval, and Pascal, have had a definite influence on Leibniz with respect to the discoveries he was about to make in establishing his calculus, and especially on the nature of the non-linear curvature of light, which clearly demonstrated the inferiority of the methods of Descartes, Newton, Euler, and later, LaPlace, and Cauchy. The papers that Leibniz published in the **{ACTA ERUDITORUM}** *on light propagation are exemplary of this polemic. The uniqueness of the approach developed by Leibniz, is that physical–space-time cannot be simplistically reconstructed on the basis of points and straight lines, but rather must be based on an axiomatically different form of geometry that must rely essentially upon-multiply connected-circular action.*

In 1672, Leibniz arrived in Paris on a diplomatic mission to Louis XIV, in the company of the nephew of the Elector of Mayence, Frederich von Schonborg. He is 26 years old, and is immediately invited to meet with Christian Huygens who had been nominated by the Prime Minister of the King, Jean-Baptiste Colbert, to be president of the Academy of Sciences a few years earlier. Leibniz is fascinated by the work that Huygens had been pioneering with the pendulum in his work **{HOROLOGIUM OSCILLATORIUM},** *and the work in* **{OPTICAL PHYSICS},** *in search of a discovery of principle relative to the phenomena of light processes that Roemer, Fermat and Huygens had been developing. Huygens became Leibniz's geometry teacher, and began by initiating him to the works of Gregoire de Saint Vincent, Pascal, Roberval, Sluse, Descartes, and others. Leibniz knew nothing of geometry or mathematics, but plunged into everything with total enthusiasm and optimism.*

However, instead of assimilating the methods of his teachers, as a typical student would have done, Leibniz discarded those he had no affinity with, such as Descartes and Newton, and used the others, such as Roberval, Pascal, Huygens and Fermat, as a starting point for his own creative invention. His method consisted in learning everything by reinventing everything.

Leibniz had a unique talent that no other human being had, in his century: he had a completely universal view of reason and of the world as a whole. As Bailly said of him in his {EULOGY OF LEIBNIZ}: *"This man was born to see everything in the large. The infinite, the being of beings, as incomprehensible as they may have been, did not stop him, and his mind plunged into the depth of these ideas!" (Jean Sylvain Bailly*, {ELOGE DE LEIBNIZ}, *Berlin, p. 5)*

{HAPPINESS AND THE PRINCIPLE OF PROPORTIONALITY}

The happiness of Leibniz was especially characterized by his discoveries of principle, which he likened to the phenomena of light reflecting the creative power of man made in the image of God; for he understood that light and reason flowed from the same fundamental principle, and that if he held that principle like a lantern, high enough in front of everyone, he would be able to enlighten the entire world by reliving the discoveries of the past and become the caustic beacon transferring them to the generations of centuries to come. The following statement, written in 1671, captures the spirit of enthusiasm that is to be reflected in the discoveries of principle that he is about to make with his calculus.

"Thus, hope and faith are founded on love, and all three on knowledge. Love is a joy of the mind arising out of contemplation of the beauty or excellence of another. All beauty consists in a harmony and proportion; the beauty of minds, or of creatures who possess reason, is a proportion between reason and power, which in this life is also the foundation of the justice, the order, and the merits and even the form of the Republic, that each may understand of what he is capable, and capable of as much as he understands. If power is greater than reason, then the one who has that is either a simple sheep (in the case where he does not know how to use his power), or a wolf and a tyrant (in the case where he does not know how to use it well). If reason is greater than power, then he who has that is to be regarded as oppressed. Both are useless, indeed even harmful. If, then, the beauty of the mind lies in the proportionality between reason and power, then the beauty of the complete and infinite mind consists in an infinity of power as well as wisdom, and consequently the love of God, the highest good, consists in the incredible joy which one (even now present, without the beatific vision) draws out of the contemplation of that beauty or proportion which is the infinity of omnipotence and omniscience."

"…From this it follows inexorably that charity, the love of God above all, and true contrition, on which the assurance of blessedness depends, is nothing other than that love of the public good and of universal harmony; or rather, on that account, {THE GLORY OF GOD AND UNDERSTANDING ARE THE SAME} *(emphasis added), and how great it is in itself to make greater, for there is no more distinction between universal harmony and the glory of God, than between body and shadow, person and picture, between a direct and a reflected ray of light, since the one is what is in fact, the other what is in the soul of him who knows it. For God creates rational creatures for no other reason but that they should serve as a mirror, in which His infinite harmony would be infinitely multiplied in some respects. From which must arise in due course the completed knowledge and love of God, in the beatific vision or the incomprehensible joy which the mirroring, and to a certain degree the concentrating of the infinite beauty in a small point in our souls, must bring with it. And thus, a burning mirror or burning glass is the natural image here." [Gottfried Wilhem Leibniz,* {OUTLINE OF A MEMORANDUM: ON THE ESTABLISHMENT OF A SOCIETY IN GERMANY FOR

THE PROMOTION OF THE ARTS AND THE SCIENCES (1671)}*, in* {THE REAL POLITICAL ECONOMY OF THE AMERICAN REVOLUTION}*, edited by Nancy Spannaus, Washington D.C., 1977, EIR, p. 215-16.]*

The writings of* {ACTA ERUDITORUM} *of Leibniz were meant precisely to be the result of a similar examination of the process of light propagation* {IN PROPORTION WITH} *the human mind; however, as if in a glass darkly, for the truth of what is to be discovered, is merely indicated by the shadows of our sense perception against the wall of Plato's cave. But, when the light of this mental process of discovery is brought to a* {MAXIMUM}*, or to an* {OPTIMUM} *level, that is, when the underlying ordering principle between the behavior of light phenomena and the development of the human mind is understood to be ordered by the same law of the universe, then this great interval of proportionality, this* {DIVINE PROPORTION OF A PRE-ESTABLISHED HARMONY} *becomes a powerful force to change the world as a whole. However, this is not a brute force, but a delicate and a decisive least action force. As Abraham Kaestner noted in his preface to Leibniz's* {NEW ESSAY ON HUMAN UNDERSTANDING}: *"The representative force, with which M. Leibniz has endowed his principles, seemed dubious even to Mr. Wolf. Yet, this same Mr. Wolf had brought in to the full light of day this truth, that the universe is a whole whose parts are so intimately connected that, one could not change the least thing without changing it into a completely different universe, that it holds together the spider's thread with the same force that pushes or pulls the planets around the sun." (Fidelio, winter 2003) It is to the understanding of such a force that the following papers exhort the reader to now fix his attention.

The following selection of 8 articles have been translated from the French book, G.W Leibniz,* {LA NAISSANCE DU CALCUL DIFFERENTIEL*}, 26 articles des Acta Eruditorum, Introduction, traduction et notes par Marc Parmentier, Paris, Vrin, 1989; and with consultation of the Latin original from, G.W Leibniz, {*MATHEMATISCHE SCHRIFTEN}, *Herausgegeben Von C.I.Gerhardt, 1962, {*GEORG OLMS VERLAGSBUCHHANDLUNG HILDESHEIM.*}

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PART I

THE PRINCIPLE OF OSCULATION: THE METHOD OF EVOLUTE-INVOLUTE.

At the center of his calculus, Leibniz developed an entirely new conception of **{TRANSCENDENTAL TANGENCY}** *which represented a new method of* **{MEASURING CHANGE}** *in physical space-time, and specifically in connection with physical processes of light propagation, such as caustics, and whose measure of curvature he called* **{OSCULATION}**, *meaning "embracing."*

By initiating this new geometric form of transcendental measurement, Leibniz revolutionized the notion of measurement in physics, and established a higher form of least-action which was aimed at replacing the Galileo-Newton linear-scalar measurement of action at a distance. Leibniz forced the whole debate on the tangency function, to be shifted away from the degenerate form of linearization in the small, and to become centered around the emergence of a very interesting series of paradoxes: the first one being, how can a straight line, which is characterized by no-change in curvature, that is, no-change in direction, be the measure which expresses constant change in direction, and constant change in curvature of the circle? How can linearity be the measure of non-linearity? How can **{NO-CHANGE}** *be the measure of* **{CHANGE}***?*

If the straight-line tangent has long been considered the physical geometric measure of direction, steepness, or even velocity of a curve, at a certain point of that curve, Leibniz will supercede this reductionist view by creating a higher form of measuring change, at the transcendental level. The idea of the **{OSCULATING CIRCLE}** *will be the* **{MEASURE OF CHANGE IN THE RATE OF CHANGE, STEEPNESS, OR VELOCITY OF NON-CONSTANT CURVATURE}**.

This means that, while the straight line tangent has always been a fictitious **{ALGEBRAIC-LINEAR}** *expression for determining the linear direction of change, a* **{NON-CHANGE}** *determination imposed upon* **{CHANGE}**, *the all embracing* **{OSCULATING CIRCULAR ACTION}** *will express the* **{TRANSCENDENTAL NON-LINEARITY OF THE PROCESS OF CHANGE}**, *a more truthful form of* **{MEASURE OF CURVATURE}**. *Thus, the introduction of the idea of* **{OSCULATION}** *by Leibniz is aimed, primarily, at breaking, once and for all, with the accepted notion, in mathematics, that measure is always reducible to straight lines. In other words, Leibniz creates the process of* **{OSCULATION}** *as a means of uprooting the most stubborn underlying obstacle to the idea of change: the underlying assumption of Isaac Newton, Leonard Euler, Rene Descartes, et al., that* **{CURVATURE}** *can be measured under the guise of linearization in the small.*

Derived from the Latin verb **{OSCULARI}**, *meaning* **{TO EMBRACE}**, *the physical geometric meaning of* **{OSCULATION}** *relates to a higher form of* **{TRANSCENDENTAL TANGENCY CONTACT}** *between curves which produces a minimum angle between them,* *which Leibniz called an* **{OSCULATING ANGLE}.** *For Leibniz, an* **{OSCULATION}** *is a transcendental means of* **{MEASURING NON-CONSTANT CURVATURE}** *of conic sections, other than the circle itself, that is, the curvature of the ellipse, the parabola, and the hyperbola, but, more emphatically, the curvature of non-algebraic curves, or transcendental curves, such as cycloids and epicycloids, evolutes and involutes, astroids, caustic curves and envelopes, catenary and tractrix curves, etc. In other words, the purpose of an* **{OSCULATION IS TO MEASURE THE DEGREE OF CHANGE OF CURVATURE, OR DEGREE OF NON-LINEARITY, IN THE VARIABLE DIRECTION OF A CURVE OR A SURFACE.}**

The two pieces of Leibniz on **{OSCULATION}: "{NEW REFLECTIONS ON THE NATURE OF CONTACT ANGLE AND OF OSCULATION {..., June 1686, and {GENERAL CONSIDERATIONS ON THE NATURE OF CURVES, CONTACT ANGLES, AND OSCULATIONS},..., September 1692,** *dealing directly with the idea of* **{OSCULATING CIRCLE},** *and* **{OSCULATING ANGLE***}, are aimed at providing a new method of measuring the curvature of conic sections and transcendental curves. The other two pieces* **{ON OPTICAL CURVES AND OTHER QUESTIONS}, January 1689, and {THE CURVE DERIVED FROM LINES}...April 1692,** *relate to the idea of* **{ENVELOPES}** *of light phenomena, and pose, in the domain of transcendental curves the Ontological Parmenides Paradox of the One and the Many, with respect to what is today called minimal surfaces, or surfaces of negative curvature, such as* **{LIGHT CAUSTICS***}.*

The reader should be aware that I have inserted in the relevant places the notions of **{EVOLUTE}** *and* **{INVOLUTE},** *whenever more clarity was required. It is important to note that* **{EVOLUTE},** *from the Latin* **{EVOLUTIO***}, is a term created by Jacques Bernoulli meaning* **{EVOLUTION}***; but, for his part, Leibniz chose not to employ the Bernoulli term of* **{EVOLUTA}***, and used instead more descriptive Latin expressions such as* **{CURVE GENERATING ITSELF BY EVOLUTION}** *(curva per sui evolutionem generans), or* **{EVOLVING CURVE***}(curva evolvenda),* **{GENERATING CURVE}** *(curva generanda), or even* **{GENERATIVE EVOLVING LINE}***(linea evolutione generans), which I have translated by* **{GENERATING DEVELOPMENT CURVE};** *or sometimes simply by* **{GENERATING CURVE}** *for* **{EVOLUTE},** *and by* **{GENERATED CURVE}** *for* **{INVOLUTE}.** *However, since the identification of* **{EVOLUTE}** *and* **{INVOLUTE}** *has passed acceptance into the English language, these are the terms we are going to use, more generally.*

Furthermore, Leibniz systematically used the ambiguous term **{LINEA}** *for both curved lines and straight lines; and, therefore, he forces the reader to make a higher judgement every time he encounters that term. Which line does he mean: the curved line or the straight line, or both? This point is absolutely crucial, because, in the reader's mind, there is rooted a very stubborn axiomatic fallacy whereby it is assumed that projective geometry, which Leibniz is making a higher general use of here, with respect to Desargues, is considered grounded on the straight line. That is an absolutely false assumption that must be rooted out. For example, the title itself {***DE LINEA EX LINEIS}...,** *which I have translated by A {***CURVE DERIVED FROM LINES}...,** *as opposed to {***A CURVE DERIVED FROM CURVES}...or {A LINE DERIVED FROM LINES}...** *is aimed at maintaining the ambiguity. In point of fact, the case of {***A CURVE DERIVED FROM CURVES}** *results in the loss of the ambiguity, because the*

generated* {CURVES} *could never include the possible case of a curve derived from straight lines. However, that case cannot be excluded. Indeed, since by virtue of Leibniz's principle of continuity, straight lines are just a particular case of curved lines, that is, the case of lines that have zero curvature,* {LINEA*} must therefore apply to curved lines, as well as to straight lines. This is another way of saying that non-linearity of curvature, as measured by* {OSCULATION}, *must supercede, but also subsume, linearization.

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1

NEW REFLECTIONS ON THE NATURE OF CONTACT ANGLE AND OF OSCULATION, AND OF THEIR USE IN APPLIED MATHEMATICS, IN ORDER TO REPLACE COMPLICATED FIGURES BY MORE SIMPLE ONES, by G.W. Leibniz, Acta Eruditorum, Leipzig, June 1686.

"When we scrutinize the infinitesimal parts of any curve, we can study not only its {direction}, as it has been done up until now, that is to say its slope or inclination, but also the variations of direction, in other words, the {curvature}. So, in the same way that Geometers have measured the direction of curves by the simplest curve having the same direction at any given point, that is to say the tangent straight line, similarly, I measure the curvature by the simplest curve having at a given point not only the same direction but the same curvature, that is to say, the {circle} which is not only tangent to the curve, but moreover, {embraces} it, which is what I will explain in a moment.

"So, if the straight line is the most appropriate to determine the direction of a curve, its own direction being identical at every point, the circle is the most appropriate to determine the curvature, because the curvature of a circle is everywhere the same. I say that a circle {embraces} a given curve, situated in the same plane at a given point, when it produces with it the smallest angle of contact. Among an infinity of circles tangent to a curve at a given point where its concavity varies, we can always determine one circle which merges more completely with it, and which clings to it longer, that is to say, to use the language of Geometry, which approaches it to the point that, between the curve and the circle, no other circular arc can be traced that could touch the curve. This minimal angle of contact between a circle and a curve, I call the {osculating angle} just like we call, {contact angle} the smallest angle between a straight line and a curve. (1) If, indeed, a straight line and a curve produce a contact angle, no other curve can come between them; for the same reason that no circular arc can come between a circle and a curve when they form an osculating angle."

"In order to find a means of getting the osculating circle, you must represent the following: just as {tangents} are generated by equations containing two equal roots, that is, by making two intersections, and just as {inflection points} are generated by the equality of three equal roots, you will generate {osculating} circles, like any other osculating curve, by establishing the equality of four roots, that is by fusing two contacts into a single one. And, if two curves have a common tangent line, the two curves will also be tangent to each other as well; similarly, the curves embraced by the same circle will also embrace each other. (2)

"That is why, just like when two concurrent straight lines meet at an intersection point, we consider that they form a rectilinear angle which is the same ordinary angle as that formed by their tangents (because the difference resides in a contact angle which is infinitely small and practically null with respect to a rectilinear angle), similarly, when the tangents to

two concurrent curves coincide, that is to say, when the two curves are tangent, we consider that at their point of intersection they form the same contact angle as their osculating circles, the difference consisting of an osculating angle, infinitely small and practically null with respect to the contact angle of the two circles. This means that an ordinary angle between two straight lines, a contact angle between two circles, and an angle of osculation (of the first degree), can be considered in the same type of relationship as a line, a surface, and a solid. In point of fact, not only is a line smaller that any surface, but it cannot even constitute a part of it, only a certain minimum, in other words a boundary.

"But if three, four contacts, or more coincide (producing 6, 8, equal roots, or more), you will therefore have osculations of the second degree, of the third degree, or more, surpassing the osculation of the first degree, to the extent that the osculation of the first degree implies a more perfect contact than the ordinary linear contact. If a circle is tangent to a straight line, it cannot embrace it, and if it happens that a circle embraces another circle, they no longer form two distinct circles, but a single one. In the other cases, a circle will be able to embrace any other curve situated in the same plane, and to generally discover what is the degree of contact, or of osculation between two curves, you must examine in how many points they can fuse into one.

"All of this is of a remarkable practical use. In mechanics, in catoptrics, in dioptrics, some brilliant consequences have resulted from the following observation: curves have the same angle or inclination or direction as their tangents; if, for example, a body is carried by a composed movement, its direction shall be that of the trajectory of its tangent, and if you leave the object to its own devise, it will pursue its course in a tangential way; similarly, an incident ray produces with its surface of contact the same angle as the tangent plane to it; and in a similar fashion, we can derive marvelous engineering procedures from the study of osculating curves.

"Have we discovered, for example, some curve or some figure with an important or useful property, but that neither the lathe, nor any other apparatus could reproduce easily; you could replace an arc of this curve (of a limited length of course, but sufficiently long in practice) by another arc, practically blending with the other one, and more easily drawn, which touches and embraces the first curve; at first glance the easiest curve to trace is an arc of a circle. It turns out that in applications of catoptrics and in dioptrics, the circle {replaces} the parabola, the hyperbola, or the ellipse, and imitates them with its own pseudo-foci.

"Take for example, a circle whose diameter is equal to the parameter of a conic section, and whose center lies on the axis of that conic section; imagine then a circle embracing that conic section at its summit, in its concavity. The circle will not be perceptibly different from the conic section, if you consider a portion of the arc, which is sufficiently small. It is for that reason that the distance between a circular concave mirror and its focus is equal to a quarter of the diameter, since the distance between the focus and the summit of a parabola is equal to a quarter of its parameter, and that the foci of the parabola and of the osculating circle coincide. And the same applies to other types of curves, taking into account the usefulness and the specificity of each practical problem.

"If we understand this, we cannot help but grasp {how such results contribute in bringing the refinements of geometry to our daily lives}. But, for the time being, I only wanted to bring to the reader's attention the idea of this reflection, so that it does not get lost. And finally, I would very much enjoy seeing that the discussions of Geometers, on this idea of contact angle, are not reduced to trifling questions, but rather that they should be transformed, by means of this approach, into solid and fruitful truths."(3)

NOTES:

(1) *The issue of* {OSCULATING ANGLE} *is very special because it is not directly comparable with the usual meanings and functions of what geometers call a* {RECTILINEAR ANGLE}, *or a* {CONTACT ANGLE}, *etc. An* {OSCULATING ANGLE} *is an infinitesimal angle, or what Leibniz called himself an* {INFINITANGULUS}; *a minimum angle which is incomparable, or incommensurable with any other angular magnitude, because the two curves are fused together by an infinitesimal difference, such that both partake of the same curvature in such a way that, by virtue of the principle of continuity, no other curve could be fitted between them. In that sense, the two curves are one and yet separated by an* {INFINITANGLE}, *which is a minimal contact that could not be generated with a straight-line tangent. A similar infinitesimal, ambiguous, and quasi-non-existent-magnitude, is also applicable to the infinitely small intervals between* {ORDINATES} *that are tangent to the* {EVOLUTE}*. The source of this new discovery of* {OSCULATING TANGENCY} *can be found in Huygens. In* {PROPOSITION 11}, *of his* {HOROLOGIUM OSCILLATORIUM}, *Huygens developed a method of determining points on an* {EVOLUTE} *by identifying them as centers of curvature of the* {INVOLUTE}. *Leibniz will see, in this normal to the curve, the radius of an osculating circle and the germ of the idea of* {INVERSION OF TANGENTS}.

(2) Jacques and Jean Bernoulli have introduced in the question of {OSCULATION}*a controversy that took completely the attention away from the essence of the question as Leibniz posed it. Indeed, Leibniz emphasized the importance of addressing the higher dimensionality of the generating process, the {embracing the curvature}, as a higher form of* {TRANSCENDENTAL TANGENCY}; *instead, the Bernoulli brothers argued that the important question was the number of times the* {OSCULATING CIRCLE} *was cutting the curve. Leibniz indicated the necessity of two coinciding circles, separated by an infinitesimal difference between them; the Bernoullis made use of only one circle, and no use of the infinitesimal. Leibniz stressed the importance of involving a short segment of the circle which merges with the curve, either on the concave side, or the convex side of a curve; the Bernoullis have stressed the importance of crossing the curve, at the point of osculation, and sometimes even at different points on the curve, and using practically the entire circle to establish tangency. Leibniz emphasized the transcendental difference between a* {CONTACT ANGLE} *and an* {OSCULATING ANGLE}; *the Bernoullis made no mention, whatsoever, of the* {OSCULATING ANGLE}. *Leibniz identifies the importance of the* {CENTER OF OSCULATION} *on the* {EVOLUTE}, *and of the* {RADIUS OF OSCULATION}*; the Bernoullis made use of neither of them. In other words, both Jacques and John Bernoulli insisted in doing something totally linear, and called it osculation, after Leibniz. The reader can*

find demonstrations of this in John Bernoulli, {DIE ERSTE INTEGRALRECHNUNG}, *Leipzig und Berlin Verlag von Wilhem Engelmann, p. 63-67, 1914.).*

In the Acta of March of 1692, Jacques Bernoulli wrote an article on {OSCULATION}, *in which he indicated his disagreement with Leibniz, by treating the intersection of a curve and an* {OSCULATING CIRCLE} *as an ordinary tangent contact. Bernoulli wrote the following:*

"The simple contact of a circle with a curve is found by the equality of two roots, and the locus of the centers of the circles are tangent to a surface; the osculating circle of the first degree is found by the equality of three roots, and the locus of the centers of these circles is a curve, the evolute; finally, the osculating circle of second degree is found by the equality of four roots...that is why I cannot quite see in what sense what Leibniz says is true, that is, that the contact is found by means of two equal roots, the inflexion by means of three, and the osculation of first degree by means of four."

Eight years after the beginning of the controversy, in the Acta of August 1694, Leibniz still maintained the same firm position:

"In ending, I must say a word about our little controversy, of a while back, concerning the number of roots pertaining to an osculation, and on the observations that the illustrious Professor (Jacques Bernoulli) had made then. Indeed, what I wrote at the beginning, when I submitted this concept to the attention of Geometers, still holds true today: when a circle embraces a curve, there is a fusion of two contacts, that is to say, of four intersections, and thus there are four equal roots (a biquadratic idea). However, it is also true that if you can find a circle that meets a curve in only three points, we can also consider it to be an osculating circle, because the fourth point does exist, even when we don't mention it. The reason is that the circle can never be construed to cut a curve whose concavity never varies, into three points of intersection, without also cutting it into a fourth point. On the contrary, in order to get a circle cutting a curve in only three points, it is necessary that, on the portion of the curve where they are, there exist an inflexion point. And, even in such an inflexion point, we may consider that four intersections do coincide with an osculating circle, that is, two circular contacts situated on the same side (sic. both sides?) of the curve, one just before the point of inflexion, the other just after; but, one in the concavity, the other in the convex part of the arc which is composed of the two portions of the curve. These contacts that are getting closer and closer together progressively will end up fusing together at the point of inflexion itself. In fact, an inflexion is nothing but the common extremity where two curves are tangent to one another, one is concave, the other is convex, (the two curves forming a single curve). So, in any osculation, there is always the coincidence of two contacts, that is, four intersections. However, if we study the intersections of a curve with a straight line, we could always consider that at the point of inflexion, there is the coincidence of three points of intersection, that is to say of one contact and one intersection, and not of two contacts."

Several years later Leibniz will admit to Jean Bernoulli that he was wrong and his brother Jacques was right. It is not clear why Leibniz admitted he had been wrong and gave up the terrain he was fighting on. Similarly, the differentiation that Leibniz made with respect to the {RECTILINEAR ANGLE} *(between two straight lines), the* {CONTACT ANGLE} *(between a*

straight line and a curve), and the* {OSCULATING ANGLE} *(between two curves), is forced into a much clearer axiomatic definition, when the debate is grounded on the terrain of* {NON-LINEAR CURVATURE VS LINEARITY}*. That is why, by creating the infinitesimally small* {OSCULATING ANGLE}*, infinitely smaller than a contact angle between a circle and a straight line, and which is by nature curved, Leibniz had established a transfinite difference of degree, like an actual difference of cardinality between a first differential and a second differential, between tangency and osculation, between linearity and curvature, a difference of species, as he says himself, like the dimensionalities that exist between "a line, a surface, and a volume".

A similar problem arises with the construction of the catenary curve and of the tractrix curve. Because of the most natural character of the* {EVOLUTE-INVOLUTE} *relationship, in this case also, the* {TRANSCENDENTAL MEASURE} *of osculation requires that the two curves be generated as a pair and not separately, as if in some kind of abstract exercise. The point is that* {OSCULATION} *implies a higher density of singularities per infinitely small area of a curvature that is everywhere changing its rate of change. Fortunately, however, Gaspard Monge, and later, Karl Gauss and Bernard Riemann, will reestablish the Leibniz conception of osculation in their applications of {spherical osculation} to envelopes, and as expressions of hyper-geometric surface contacts.

(3) The technical difficulties of polishing non-spherical lenses and mirrors for astronomical instruments represent some of the most important areas of practical application of the Leibniz* {OSCULATING PRINCIPLE*}. For example, one of the problems that Leibniz was trying to solve was to calculate the appropriate osculating circle for polishing a particular case of parabolic mirror.

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2

ON OPTICAL CURVES AND OTHER QUESTIONS, by G.W. Leibniz, Acta Eruditorum, Leipzig, Jan 1689.

"It is while taking a long trip in the service of my Prince Serenissim, and during which I have had the opportunity to scrutinize here and there some documents from different Archives and Libraries, that a friend has brought me the Acta of Leipzig which permitted me to know what was happening in the Republic of Letters, without having access to the latest publications.

"When going through the issue of June 1688, I have stumbled upon the summary of the {Principia Naturae Mathematica} of the celebrated {Isaac Newton}, and although he was addressing problems that were quite remote from my preoccupations at the time, I have read it with avidity and great pleasure. Indeed this man is one among a handful of men who have succeeded in pushing the limits of science, as would show, by themselves, the series that Nicolas Mercator of Holstein had gotten by way of division, but that Newton, with an even more considerable invention, had succeeded in adapting to the extraction of pure as well as affected roots. While we are on this subject, I have imagined while waiting for the method of series, besides the transformation of irrational curves into rational curves of the same measure (I call rational the curves from which one can always express the ordinates with respect to the abcissas, by means of rational numbers), a process for given transcendental curves, and through which the issue of extraction no longer exists. Indeed, I take an arbitrary series, and then I submit it to the conditions of the problem, while I identify the coefficients.

"But I expect the most brilliant discoveries from this last work of Newton, and if I may judge by the summary of the Acta, I have to admit that if, on the one hand, he communicates many new results of great importance, on the other hand, he also tackles a certain number of problems that I have occupied myself with; aside from the question of causality relative to celestial movements, he has also worked on the explanation of catoptric (reflection) and dioptric (refraction) curves, as well as on the resistance of different media. Descartes has known about such {Optical Curves} but, he has not whispered a word about them to anyone, and his commentators have not found any traces of them. The whole matter, in point of fact, has nothing to do with ordinary analysis. I know that, later on, Huygens has also made their discovery (but he has not yet communicated his results either), and now, it is the turn of Newton. As for myself, I have also discovered them, but by a different route. Even though I was familiar with general methods of approach, it is the remarkable discovery of our dear M. Tschirnhaus, which was published in the Acta, and where he treated entire curves as foci, which gave me the idea of discovering the proper and very elegant methods required. I shall explain this process by an example, which should clarify everything else.

"Given a point A and a curve BB, on which are reflected rays AB, find the curve CC which will reflect, a second time, the rays ABC which will converge on a common point D.

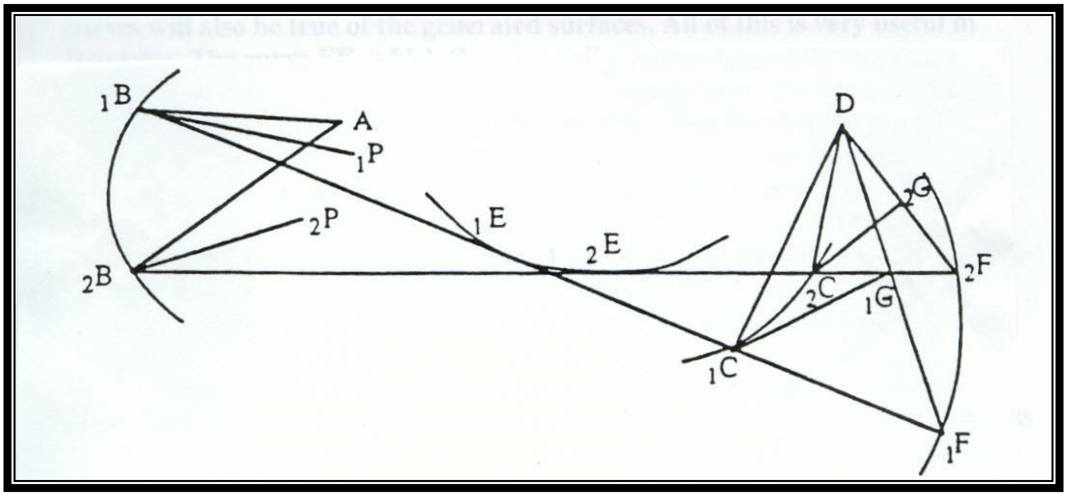

Figure 1. Least action principle of light propagation.

"Here is the solution I have found on my first attempt. The curve BB being given, it is clear that the focus-curve EE is also given from point A, and by means of that curve; thus, two conjugated foci being given, curve EE on the one hand, and point D on the other hand, it is clear that curve CC can be found whose two foci are EE and D; this curve CC is the curve we are seeking. But, there exist better ways of constructing this. In fact, A1B + 1B1E + arc 1E2E = A2B + 2B2E and D2C + 2C2E + arc 2E1E = D1C + 1C1E, consequently the sum AB + BC + CD is always equal to a constant straight-line segment. (1) If, at the same time, a thread is enrolled around curve EE, and is connected to point D, and if we generate the curve of evolution EE (the evolute), a marker which is held at the end of the extended thread will trace curve CC (the involute). If, on the contrary, the same thread is fixed by its other extremity to point A, the marker, which is extended from it, will draw curve BB. But, if curve EE were to disappear, the simplest construction would be the following: take away from a constant segment (equal to AB + BC + CD) the given segment AB, then take the segment BF equal to the difference, and trace it in a manner such that it makes an angle FBP equal to ABP, with the normal PB to the curve (or to the tangent of the curve) BB. Trace the normal GC to point G through the middle of the straight line DF, such that it cuts BF at point C that is the sought for curve; you can see that GC is tangent to the curve CC.

"If you rotate that figure around its axis AD, what I have said about the curves will also be true of the generated surfaces. All of this is very useful in Dioptrics. The curve EE, which the rays strike without going through any reflection or refraction, is what I call the {Acampe} curve. There also exist {Alcaste} curves, which reflect rays without refracting them. Such are the generative processes of curves described by way of simple development of the caustic curve EE, a process that Huygens was the first to study, but with another purpose in mind. We get curve FF by locating CF (as an extension of BC) equal to CD. We would get the same result, taking into account the specificity of each problem, if we were to replace

point A and D, or only one of them, by foci-curves, or if the point were at infinity, in the case of parallel rays.

"What I wrote in my article on the resistance of media, I had already discovered, in Paris, twelve years ago, and I had communicated certain elements of it to the illustrious Royal Academy. (2) Finally, because I have also had ideas on the subject of physical causality with respect to celestial movements, I decided that it would be good to communicate some of them, and write them up in an article. I had made up my mind to not publish them before I had more time to confront with more scrutiny the required geometric laws against the more recent astronomical observations, but (aside from the fact that completely diverging tasks are keeping me away, and do not permit me to accomplish this) the work of Newton has prompted me to make public everything that I had discovered, in order to spark the light of truth by means of the confrontation of methods, and to also bring forth a contributing aid from this great penetrating mind."

NOTES.

(1) The fact that the sum of AB + BC + CD (Fig. 1) is constant, demonstrates that the propagation of reflexive light, in the same medium, follows the isochronic law of the **{LEAST ACTION PRINCIPLE};** ***that is, it follows the most determined, easiest, and*** **{SHORTEST PATHWAY}.** ***Pierre de Fermat similarly demonstrated that, in the case of refraction, the propagation of light proceeded in the*** **{SHORTEST TIME}.** ***The Fermat-Leibniz demonstrations represented the most devastating*** **[DISCOVERY OF PRINCIPLE OF LEAST SPACE-TIME}** ***against the Cartesians and Newtonians who rejected the new discovery out of hand. In the Leibniz construction, since point A and mirror B are given, the pair of Incidence ray (A1B) and reflective ray (1B1E) form equal angles with respect to the normal P1. Since point D is also given, with respect to curve EE, the sought for curve CC can also be found by inversion of the same angle values for incidence ray (1E1C) and reflective ray (1CD). If light is constantly well focussed at A, as well as D, then, the pencil of rays, ordered in position, shall form a surface which is everywhere*** **{ISOCHRONIC},** ***that is to say, in which each ray shall represent the shortest possible pathway of light propagation which is to travel from A to D in the same time. Thus, by virtue of the*** **{LEAST ACTION PRINCIPLE},** ***the ancient Greek rules of equal angles (reflection), and of constant ratio in the sines of angles (refraction), are proven to be universally true for the first time. In other words, it is the*** **{LEAST ACTION PRINCIPLE}** ***which determines and establishes the rule of angles, and not the other way around. As a result of this discovery, Leibniz became the first to establish a calculus of Optical Physics, by creating a direct correlation between*** **{EVOLUTES}** ***and*** **{CAUSTICS}.** ***This revolutionary discovery was further developed, three years later, in 1692, by Jacques Bernoulli in his article*** **{LINEAE CYCLOIDALES EVOLUTAE, ANTEVOLUTAE, CAUSTAE, ANTICAUSTAE, PERICAUSTICAE} (CYCLOIDAL EVOLUTE CURVES, ANTIEVOLUTES, CAUSTICS, ANTICAUSTICS, PERICAUSTICS).**

(2) It is during this period of January 1689 that Leibniz wrote in Vienna his first articles on optical physics, in which, for the first time, he based optical phenomena on the Fermat **{LEAST TIME PRINCIPLE}.** ***The new principle is discovered from the idea of final causality whereby,***

in opposition to the pull-me push-me false mechanical causality of Descartes and Newton, Leibniz established that the pathway of light in nature will follow an **{INTENTION}** *that is not only the shortest, and the most timely, but also the easiest and most convenient that nature will take to arrive at its end. In Leibniz's mind this was also coupled with the central problem of studying optical curves for the practical purpose of constructing microscopes, telescopes, corrective lenses, etc. The purpose was very much soundly grounded in an economic machine-tool principle that the Colbert Academy of Sciences was task oriented to discover and apply to new technologies. The revolutionary discovery of Ole Roemer in the determination of the speed of light was a classical example of anomaly disproving the Cartesian conception of simultaneity of light propagation. The further demonstration of the* **{LEAST ACTION}** *principle of the* **{SHORTEST TIME}** *refraction of light by Fermat, and the Huygens discovery of the spherical propagation of light, and the* **{TAUTOCHRONIC}** *quality of the ordinary* **{CYCLOID-PENDULUM CURVE}**, *as the curve of same time, were all crucial contributions to the seminal breakthroughs of Leibniz. The discovery of Huygens'* **{TAUTOCHRONE}** *was crucial to show the shortest time feature of the cycloidal pendulum, as a first step towards understanding the isochronic feature of light in the* **{ISOCHRONE CURVE}** *of Leibniz, and in John Bernoulli's discovery of the* **{BRACHISTOCHRONE}.**

There is also in this **{OPTICAL CURVES AND OTHER QUESTIONS}** *a motivation on the part of Leibniz to approach pedagogically the problem of optical curves, in a similar fashion as in his* **{CURVE DERIVED FROM AN INFINITY OF LINES}**, *where the problem is not at all oriented toward the discovery of a particularity pertaining to a given curve, but rather to discover a curve which must have a certain property, or must satisfy a specific universal and epistemological condition. The reader will notice that, for that purpose, Leibniz will bring the attention of the reader to bear especially on what Tschirnhaus called the* **{FOCUS CURVES}** *of caustics, which will lead him to the investigation of the* **{ENVELOPE}** *of a family of curves and the* **{CURVING OF CURVATURE}** *which became the explicit topic of* **{A CURVE DERIVED FROM AN INFINITY OF LINES}.**

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3

CONSTRUCTION OF A CURVE DERIVED FROM AN INFINITY OF LINES WHICH ARE WELL ORDERED, CONCURRENT, AND TANGENT TO IT; AND A NEW APPLICATION FOR THE ANALYSIS OF INFINITIES, by G. W. Leibniz, Acta Eruditorum, Leipzig, April, 1692.

"It is customary in geometry to call {determined ordering} the family of parallel lines, however numerous, and traced between a curve and a straight line (the {directrix}); when they are normals to the directrix, (which plays the role of {axis}), we identify them as {well ordered}. {Desargues} generalized this by also considering as ordered in position, those straight lines, which {converge} toward a common point, or {diverge} from it. So, it is also admissible to include parallel lines between the converging and diverging ones, by representing their common point at infinity.

"There are many other ways of conceiving of an infinity of lines traced according to some common law, without them being parallel, or convergent to a single common point, or diverging from it; it is such lines that I will name {directed ordinates} or lines given in ordered position. If a certain mirror, for example, or rather a plane section of a mirror, taken parallel to its axis, reflects the rays of the sun coming from a certain figure whose position is given, arriving either directly, or after an additional reflection or refraction, then the reflected rays will form an infinite number of straight lines in ordered position; and for each point you designate on the mirror, a corresponding reflected ray will be given (among the other rays subsisting at the same time).

"However, I consider as given in ordered position (ordinatim ductae) not only straight lines, but also all sorts of curved lines, provided they are given according to a {lawful} mode in which for every point of a certain curve (as ordered in position), a line can be drawn at a corresponding point. The lines so ordered reproduce themselves in order of succession as we traverse the point of ordinate (for example the line whose rotation around the axis generates the mirror surface referred above). Now, although these lines do not all concur in a common point, nevertheless, if you take two such lines {very close} to each other (with an infinitesimal difference, in other words having an infinitely small distance between them), they will run concurrently in ordered position, in such a way that a point of concurrence can be assigned, and these points of concurrence, taken in order, reveal a {curve of concurrence} which is the common locus of all concurrences between the closest lines, and is remarkable by the fact that it touches all the directed ordinates, and is formed by their concurrences (1); it is not necessary to demonstrate this property since it is clear enough for any one who wishes to concentrate on this."

"Such is the case of {the generating development curve} ({evolute}), which is tangent to all the perpendiculars of the curve generated by that development ({involute}), following the discovery of {Huygens}. Such, also, are the numerous curves {generated by co-developments}, which Master D.T. (Tschirnhaus) invented, such as the {quasi-focus} introduced by the same

author; when the concurrence of the rays does not form a point, but instead a region, {the focus} is a {curve}, formed effectively by the overlapping of dual concurrences of neighboring rays. But because this process is not reducible to straight lines, one should understand that we are dealing here with something that is also taking place within curves (2)."

Figure 2. Illustration of a caustic of light burning through cold water.

"We have described in the Acta ({ON OPTICAL CURVES AND OTHER QUESTIONS}, 1689) the construction of a reflexive curve which, reflecting according to some given law, would project rays emanating from any light source, passing through any transparent medium, or being reflected through a system of different lenses or mirror, and would permit you to reestablish such rays once more convergent (divergent or parallel) in a well ordered way. Such a curve can be formed by the intersections of an infinity of ellipses, (hyperbolas or parabolas). And, this is the way to discover the method by which one can solve the following difficult problem: Indeed, since an infinite number of ellipses can be given in an ordered position, the curve of their concurrence can also be found. This method opens the door to great many new discoveries, which, without it, would otherwise remain inaccessible; that is the reason why I wanted to open this new road for Geometers. (3)

"So, all of this is based on my {analysis of indivisibles} and the calculation that this method requires is nothing else but my differential calculus. Indeed, once we have established a particular equation (which corresponds to a particular curve in a given family of curves given in order), which is general (because it shows the law common to all of the curves), if we then seek its differential equation, I will further show that those two equations will give us the sought for curve.

"However, if we are looking for the tangent to a certain curve in one of its points, it is sufficient to {differentiate the equation} of the curve, or in other words to seek the equation which is differential to the particular equation of that curve. In such a case, the {parameters} or the {constant} segments involved in the construction of the curve, or in the calculation of its equation, and which we are accustomed to designate by a, b, etc., are considered unique or {indifferentiable}, just like the tangent line itself, or several other {functions} depending upon

these, for example the verb. gr. perpendiculars to the tangent drawn from the axis to the curve. In truth, both the {ordinate} as well as the {abcissas}, customarily denoted by x and y (and which I am accustomed to call the {coordinates}, because one is ordered by one side, the other by the other side of the angle formed between the two directions taken together, are {paired}, that is, {differentiable}. (4)

"Now, in our present calculation, what is sought is not an arbitrary tangent of some curve, in some arbitrary one of its points, but rather a unique tangency for the infinite successively ordered family of curves, defined for every corresponding curve; for this reason, when we seek the point of contact corresponding to any given one of these curves, the contrary case occurs, and both x and y (or any equivalent function corresponding to the point to be determined) are {unique}; and one function must certainly be {double} or differentiable, at least one of the parameter a or b, so that while varying, we may go from one ordinate to the other within the ordered family. And surely, it is possible that several curves depending upon one parameter become constant straight lines (for example any ellipse, and most hyperbolas, have two such lines, while the circle and the parabola have only one unique one), while nevertheless the problem must ultimately be derivable from the given data, in such a way that only one unique {identified variable} (instead of several) remains {constant} (for one and the same curve); in the opposite case, the mode of ordering is not sufficiently determined.

"Nothing prevents us, in many cases where several determining equations are given, to consider several parameters as differentiable, because several differential equations can be taken to determine themselves. And most of the time, a {most constant} parameter is given (or more than one), or a parameter common to all of those directed ordinates; and for this reason, the letter designating this parameter remains indifferentiable in the differential calculus. From this it manifestly follows that one and the same equation may have many differential equations and may be differentiable according to various modes, as required by the purpose of the investigation. And, in addition, I have found that several modes of differentiation may unite together in one and the same equation.

"All these things could be explained more clearly, and be illustrated by examples, if my purpose was to make an exposition of my new method of {analysis of infinities}; but this is not the place, nor the time, for me to do so. As for those who have understood my previous articles, and who wish to meditate further on them, they will arrive without difficulty at the same results, and certainly in all of the more agreeable fashion, since they will be under the impression of making the discoveries for themselves.

"I sometimes employ {new words}, whose meanings however are explained by the context; it is not my habit to innovate in language without due reflection, unless the gains obtained are evident, not only in terms of brevity of presentation (because in fact it would hardly be possible to translate this without complicated calculations), but also by providing a certain anticipation, exciting the process of thought, and allowing the mind to penetrate matters in their universality."

NOTES.

(1) The* {CURVE OF CONCURRENCE} *that Leibniz is generating here corresponds to what Gaspard Monge will call an* {ENVELOPE}, *a century later at the Ecole Polytechnique. The conception was first established by Desargues in his* {BROUILLON PROJECT D'UNE ATTEINTE AUX EVENEMENTS DE RENCONTRES DU CONE AVEC UN PLAN}, *in which he described a surface envelope generated by a moving straight line around a fixed point. The idea of* {DIRECTED ORDINATES} *also comes from Desargues who wrote, in his theorem of perspective: "To establish what we mean when several straight lines are said to be either parallel or concurrent to a same point, we say that all of these lines have between them the same* {ORDINANCE}, *from which we conceive that from one, or many of these lines of either types of positions, they are all directed toward the same region." (In R. Taton,* {L'OEUVRE MATHEMATIQUE DE G. DESARGUES}, *p.100.) With respect to his own principle of continuity, Leibniz also stated, in a letter to Galloys, that he was also following Pascal, a student of Desargues, in this ordinate process: "Indeed, Mr. des Argues and Pascal have been quite correct in using ordinates, in a general way, as convergent or parallel lines, especially since parallels can be considered as a sub-species of convergent curves, whose point of concurrence is at infinity." Leibniz also extended this application of* {ORDINATES} *to curved parallel lines, or curved convergent lines.

(2) This is the most precise and rigorous description of a* {CAUSTIC}, *as a locus of* {INCREASE DENSITY OF SINGULARITIES} *generating a change of manifold that ever was undertaken in constructive synthetic geometry. It is also a most appropriate description of the creative process of discovery itself. Leibniz is clearly describing the mirror effect of the creative process from which an* {EVOLUTE CURVE} *is actually generating an* {INVOLUTE CURVE}; *and more significantly, he establishes the actual locus of formation of the* {CAUSTIC OF INCREASE DENSITY OF SINGULARITIES}, *as the* {EDGE OF EVOLUTE-INVOLUTE INVERSION}, *or an* {EDGE OF CAUSTIC INVERSION}, *from which a new and higher order of curvature is generated. As Leibniz said: "For God creates rational creatures for no other reason but that they should serve as a mirror, in which His infinite harmony would be infinitely multiplied in some respects. From which must arise in due course the completed knowledge and love of God, in the beatific vision or the incomprehensible joy which the mirroring, and to a certain degree the concentrating of the infinite beauty in a small point in our souls, must bring with it. And thus, a burning mirror or burning glass is the natural image here." (Op. cit., p.216.) This inversion is exemplary of the process of transformation and* {AXIOMATIC CHANGE} *which occurs when a discovery like the* {LEAST ACTION PRINCIPLE} *is added as a new measure of change in the curvature of physical space-time. The discovery adds a new degree of freedom, as LaRouche has shown to be in the case, in a Riemannian surface function, and increases man's power over the universe, proportionately. The Leibnizian method of discovering new curves, from the property of its tangents, is a similar kind of inversion, where the discovery comes from simply* {PUTTING THE CART BEFORE THE HORSE}.

(1) This is a problem that Huygens was able to solve by discovering the* {ASTROID} *as the* {EVOLUTE} *of an infinity of ellipses.

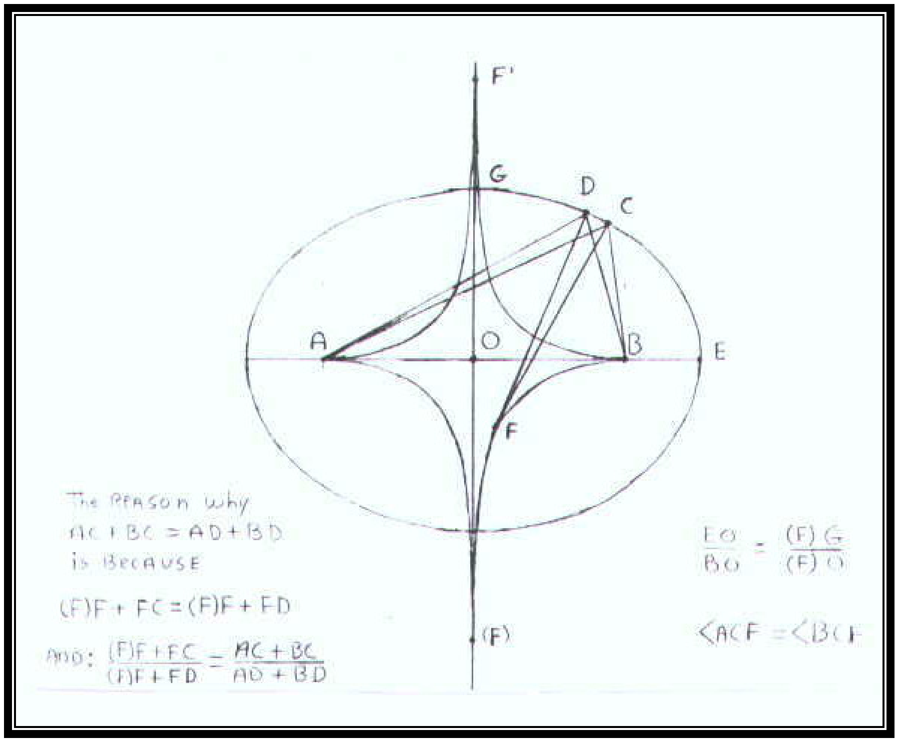

Figure 3. Illustration of Huygens' astroid as evolute of an ellipse.

(4) The references to **{ABSCISSAS - ORDINATES}** ***are a loose reference to the Cartesian coordinates that Leibniz cannot accept because of their reductionist and unreal fixity, that is, because of their undifferentiability.***

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4

GENERAL CONSIDERATIONS ON THE NATURE OF CURVES, CONTACT ANGLES AND OSCULATION, ON EVOLUTES AND OTHER NOTIONS RELATING TO THEM, AS WELL AS SOME OF THEIR APPLICATIONS, by G. W. Leibniz, ACTA ERUDITORUM, Leipzig, September 1692.

"Nothing has given me more pleasure than to see Eminent Men judging my varied attempts as worthy of pursuing through their own works; and for that reason, I have been very pleased with the article on osculations published by the illustrious professor Bernoulli from Bale in the {Acta Eruditorum} of last March. I must recognize that if he generally approves of my views, he nonetheless thinks that the foundation of some of them should be modified. However, I am so far from objecting that, quite to the contrary, each time I am taught something, I can only find something beneficial; I have deemed it necessary to reexamine the question, while I am ready to change my views if, alerted by the objections of this great scientist, I were to realize I have to do so.

"I have stated that, for me, a {contact} includes two coinciding intersections, an {osculation} involves several coinciding contacts, such that the osculation is of the first degree when two contacts or four intersections coincide (a biquadratic function), that the osculation is of the second degree when six intersections or three contacts coincide etc., and that the osculating circle, that is to say the greatest or the smallest of the tangent circles, at a point on the inside or outside of a curve (or whatever circle that approaches the curve the most) will constitute the measure of curvature, and will determine the minimum angle of contact in such a way that between two tangent curves, the angle is the same as the one formed by the osculating circles at that same point. As for curves which can have more intersection points with the circle, higher osculations can be applied, when all of the intersections are joined into a unique point; thus, sometimes, when you have a maximum or minimum of curvature, that is to say passing from an increasing to a decreasing curvature, or the opposite, two osculations coincide, that is four contacts and eight intersections.

"Furthermore, I have observed that the center of the osculating circle of a given curve is always located on the generating development curve ({evolute}) which generates another curve ({involute}) by means of unraveling a thread, and that there exist only one perpendicular (in its series) which connects the center of the osculating circle to the generating curve ({evolute}); that is to say, to the unique curve, {unique}, in the sense of the uniqueness of a minimum or maximum curvature that can be generated from a point of extension to a curve. In fact, starting with other points situated in the concavity of the generated curve ({involute}), one can develop several perpendiculars, at least two in a series of maxima or minima, {two lines unique in the series} drawn to the generating curve ({evolute}). And, since it is clear that, depending on the length of the thread that we are developing, we are describing a family of curves ({involutes}), I noted earlier that the curves that M. Bernoulli recently called co-described were all {parallel} to each other, meaning that, everywhere, one curve is at the same distance to another (the distance being everywhere

equal in the smallest interval, in other words, equal to the shortest straight line that would go from one curve to the next), or again, where a perpendicular line to one curve is also perpendicular to another, (which is my definition of parallelism in general). I have noticed that, in his solution to the problem of the catenary curve, the initial founder of these processes of developments, the illustrious Huygens, is also in agreement with my use of this measure of curvature and of my treatment of generating development curves ({evolutes}).

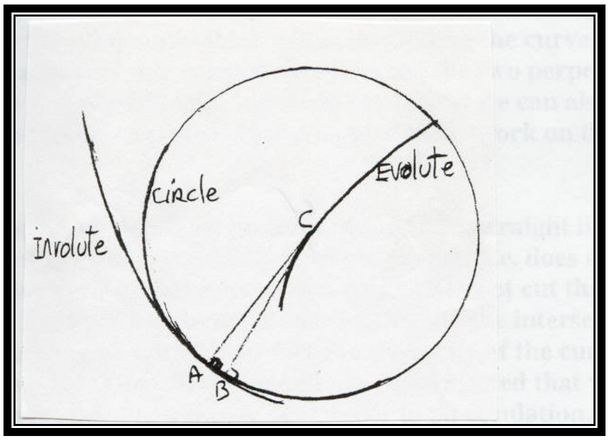

Figure 4. Illustration of an osculating circle.

"Furthermore, when there is a coincidence of three intersections between a circle and a curve, I have identified that as an {inflexion}, which means that we have the junction of a contact and an intersection. Similarly, the coincidence of five intersections brings together the junction of a contact and an inflexion, or of an intersection and an osculation of the first degree, while seven coinciding intersections would bring together an inflexion and osculation, or an osculation of the second degree plus an intersection. Thus, this makes intelligible the possibility of bringing together, however dense they might be, any number of coinciding intersections with respect to contacts, inflexions, and osculations (i.e. increase in the density of singularities). However, in a contact as well as in an osculation, the straight line or the circle are tangent to a curve by remaining either inside or outside of it, while in the case of an inflexion, they are tangent to two portions of the curve, both internally and externally, and when we take the two united together, they are no longer tangent to the curve but they cut across it. (1)

"The reason the generating development curve ({evolute}) is the locus of all of the centers of osculating circles of a given curve ({involute}) appeared to me to warrant the following explanation: given two points A and B on a curve ({involute}), the intersection C of the perpendiculars to the curve in A and B will give us the center of a circle which, if it has AC as radius, will be tangent to the curve in A, and tangent in B, if it has CB as radius, but if A and B coincide or are separated by an unassignable distance, in other words in a point where the two perpendiculars are concurrent, the two contacts coincide and the two tangent circles are fused into one which will be osculating the curve ({involute}). So, it is exactly by means of this concurrence between the two perpendiculars, whose difference is unassignable,

i.e. infinitely small, that we can also find the development curves ({evolutes}) that Huygens used in his work on the Pendulum."

"Furthermore, a circle whose center is located on a straight line perpendicular to an arc whose concavity does not change (i.e. does not become convex), and is going through the intersection point, does not cut that arc but is tangent to it. That is why if the circle cuts across the arc, the intersection is necessarily an inflexion point, and therefore the concavity of the curve does not remain on the same side. Thus, M. Bernoulli has rightly noted that when a simple intersection is added to a simple contact, or to an osculation, that is to a multiple contact, the contact is changed into a section; however, it becomes evident that when a circle embraces a curve, there exists, as a general rule (with the exception of the inflexion point), a coincidence of four intersections or two contacts; such is the case of an osculation of the first degree, to the extent we identify, as such, the ordinary osculation by means of two circles, which are susceptible to apply everywhere along the curve, that is, by fusing them into a single circle which measures its curvature by approaching it the most.

"We can say, in a general manner, that the number of intersections of a circle with another curve is normally even. Therefore, I cannot see how an osculation of the first degree could be made up of three intersections, such that an osculation of that type which had three roots would be the rule for the curve as a whole, while the osculation of four roots, involving four merged intersections, would be considered secondary and singular, which would occur only at certain determined locations. The matter is quite the opposite; each osculation involves ordinarily four intersections, that is two contacts; and it is only in the extreme case of an inflexion that the osculation, dying or being born, so to speak, would produce three intersections. Furthermore, I never intended to make a particular degree of osculation the case of three intersections, simply because the contact, for which an osculation is the most perfect expression, has degenerated by loosing one of its intersections.

"For the same reason, even for the case of higher degrees, an osculation requires, by nature, an even number of roots, and is reduced to an odd number only in the case of an inflexion point. In fact, when on top of a contact, at a given point, the circle cuts the curve, again, in two other points; these intersections must approach one another and finally merge together with the contact point, as long as the center of the circle is moving continuously. Each of these intersections must inevitably reach that point simultaneously, for if one, or the other, were to reach that point separately, and if that first circle were to become the closest to the curve, that is the osculating circle, the result would be that, since the two intersections had come to coincide with the contact point at different times, there would exist different circles which would be closest to the curve, that is to say different osculating circles, coinciding with the same given point, which would be impossible; unless, naturally, the curve were to cut itself at that point, in which case it would be playing two roles, and the circles would then embrace, in reality, two curves, even if they were to be two portions of the same curve, but such is not the case here.

"We can easily deduce from the preceding that if a circle, aside from having an internal contact, can again cut the curve (from one side to the other), in the case of an osculation (where the two intersections are joined at the contact point), the osculating circle

goes on the outside of the curve, and conversely, an external contact joins the two other intersections which gives rise to an internal osculation, and thus, when you go from a contact accompanied with an intersection, to an osculation, the circle passes to the other side of the curve.

"We must emphasize also that a {minimal curvature and a maximum obtuse opening} are created at a point of inflection, and M. Bernoulli has correctly established that, in that case, the osculating circle degenerates into a straight line; its radius is actually infinite, and its center falls at the intersection of the evolute and its asymptote. Since, in the interval of passing from diverging to converging, the two very close perpendiculars (which, up to that point, were intersecting one another on one side of the curve) now had to become parallel lines before intersecting, again, on the other side of the curve. In such an infinitesimal interval, the intersecting point of those perpendiculars had to be at infinity.

"We can find another case, however, where the generated curve ({involute}) has a {minimal curvature} and a {maximum obtuse opening}, not absolutely, but relative to a portion of an arc which maintains the same concavity; that is, with respect to a certain progression. This occurs when the generating curve ({evolute}) is such that its development cannot go beyond a certain point, nor unravel further its generating thread; such as the case when the generating curve ({evolute}) touches the two curves of concavity at the point of their tangency. (This should be illustrated with the case of the evolute of an ellipse.) We shall similarly obtain a {maximum curvature relative to a minimum obtuse opening}, when the curvature, which is increasing, after having decreased, is generated by the generating curve ({evolute}), that is, when the generated curve ({involute}) is not developed from between the two tangent generating arcs of opposite concavity, but is developed from outside of their angle. In neither case is the generated curve ({involute}) actually produced by a further extension of the unraveling thread. (2)

"I insist in making these remarks, all the more readily, because they illustrate the nature of these curves in general, and they not only give me the opportunity to put an end to the famous controversy concerning the nature of the contact angle, but also the opportunity to transform a trifling logomachia into concrete and potent applications.

"I see that, recently, while defending his thesis against M. Lagny, and while making use of the diameter of the umbra of the Moon during an eclipse, M. Eisenschmid has mentioned the use of the diameter of an osculating circle with respect to the elliptical shape of the Earth; that is, the diameter of a circle which generated an osculating angle (the smallest of all contact angles), relative to the ellipse and thus, approaching it as much as possible, with the aim of defining, from the different proportions between the diameter of the umbra and that of the Moon, the exact form of the terrestrial globe. I leave to others the care of discovering what useful results may come out of this. (3)

"I had finished this article when I found before me the Acta of the month of May, in which I read some new things by M Bernoulli, cycloidal curves, evolutes, involutes, caustics, anti-caustics, peri-caustics. I took pleasure in examining the very elegant property that he has discovered, concerning a curve, which, with straight lines converging on a given point, made a

constant angle, which was not a right angle. I also see there other remarks, which shed some light on the nature of curves more generally. We have therefore made considerable progress in the theory of curves, on the one hand thanks to the explanation of the notion of {curvature}, and on the other hand, by the use of {rolling} and of {development} to generate them. It seems to me that the intimate nature of {flexion}, that is of curvature, has given us, in part, its secret when I discovered {a measure for the contact angle}, by means of the {osculating} circle to the curve, that is of the circle which is approaching it the most and which has the same curvature as the curve at the point of osculation, as I have stated before, and as I have just repeated it.

"As far as {rolling} is concerned, it is apparently Galileo, who was the first to meditate on the curves generated in that fashion, and who first began the study of the {simplest} among them, the {cycloid}. Described in space by the drum of a wheel rolling on a plane, the many properties of this curve were demonstrated by a number of scientists. The Danish, {Roemer}, made famous primarily as an Astronomer, has discovered, I am told, while he was working at the Royal Observatory of Paris, the elegant properties of a superior {cycloid}, which is generated when a circle, or a wheel rolls on another circle, but I have not seen any of this. (Note on Roemer's {cycloidal gear-wheels}) Recently, Newton has produced on Cycloids, some admirable and universal works.

"{Huygens} was the first to produce these curves by means of {development}. {Tschirnhauss} pursued the same idea by adding what he called {co-development} curves, and by noting that such co-developing curves can be considered as {foci} and are also generated by intersections of rays; he had studied particularly the caustic, which is formed by the reflection of parallel rays on a mirror. Starting from there, I have gone much further, and I have discovered how to use them to solve optical problems (which had been the main objective of this [[speculation), and to discover {optical curves} which permitted to make the light rays converge to, or diverge from, a given point, or even to make them parallel to each other. This is what Newton in his {Principia}, and Huygens in his book on Light, have also produced, by means of different processes. I have also noted that the same process generates Acamptes figures, which do not reflect the sun rays, even though they are polished and opaque, and the Aclastes, which are transparent and constructed in a way that could refract, because of their shape and position vis a vis the Sun, but will let the rays go through without refractions. {Bernoulli} had added to this some curious considerations. Also, {Huygens} in his Treatise on Light, and {Tschirnhauss} in the Acta, had both noted that this caustic, formed by the reflection of sunrays on a concave spherical mirror, is also a cycloidal curve formed by the rotation of a circle on another circle. Finally, I have, myself, proposed recently a {new process for generating curves by the concurrence of well ordered curves}, while up until then, the process was reduced to intersections of rays, in other words, of straight lines; this process appeared to me to be quite useful for discovering the solution of a certain number of problems in a remarkable way.

"The recent developments of the illustrious {Bernoulli} with respect to the curving of a sail in the winds appear to me to be concealing a number of admirable things, about which I would not dare pronounce myself since they involve many delicate adjustments that my other tasks have not permitted me to ponder with all the attention that they require. The practical

applications that can be derived from my logarithmic measurement of {loxodroms} are numerous; however, I have difficulties in finding what measures the pathway that determines longitudes. When we consider the rigorous geometric drift of a ship, one should consider not only the form of the sail, but also that of the ship. In short, I wish to recognize all of the progress that he, and his brother, have accomplished by way of my calculus, and I congratulate them for doing as much as I would have done myself.

"But, I would like to know if they have gone beyond the point that I had reached; because, if I were to expect that from anyone, it would have to be from their respective genius. So, I would be filled with extreme joy, if they had accomplished that, particularly because I no longer have the time to research these matters with as much attention as I used to.

"However, I did not have any serious difficulty in solving the following problem: Find the curve whose elements of elements of abcissas are proportional to the cubes of the increasing of the elements of ordinates; which is perfectly realized with the catenary, or the funicular curve. But, since the {Bernoulli} have already noticed this, I will only add the following; that is, if, instead of the cube of the elements of ordinates, you were to square them, the curve in question would be logarithmic; and if the elements of ordinates were themselves simply proportional to the elements of elements, that is, proportional to the second differences of the abcissas, I have discovered that the sought for curve would be nothing else but a circle."

NOTES.

***(1) We must remind the reader of the ambiguity that led Jacques Bernoulli to misunderstand Leibniz's intention and divert the attention away from his higher conception of* {OSCULATION}*. The situation becomes more evident here, when Leibniz makes it explicit that the only time where a curve is cut by an* {OSCULATING CIRCLE}*, is when the curve is an inflexion curve, that is, when a concave curve becomes convex. This implies that if a curve, which does not change its curvature, were to be cut, it would not be cut by an* {OSCULATING CIRCLE}*, but by an* {UNASSIGNABLE SINGULARITY}.**

In the case of an* {INFLEXION CURVE}*, the* {OSCULATION} *cannot take place, but must make an* {AXIOMATIC LEAP TO INFINITY}*, from the inside to the outside of the curve, or vise versa. This is a case that pertains to the Leibniz* {PARADOXICAL PRINCIPLE OF CONTINUITY}*. In other words, for Leibniz, the* {OSCULATING CIRCLE) *is internal to the curve, that is, it embraces the curve in the concavity. If concavity inverses itself, as in the case of the* {INFLEXION CURVE}*, the* {OSCULATING CIRCLE} *must be rotated on the other side of the curve and embrace the convexity at the same time, and paradoxically, it must embrace the curve on both sides at once. An osculating circle cannot cut the curve otherwise. When such a case does occur, then, a change of species among {osculating} degrees occurs in the {transcendental domain}, just as a* {CRUCIAL SINGULARITY} *is generated,* {IN THE ALGEBRAIC DOMAIN}*, when you consider passing from the inside of a circle to the outside: an inscribed polygon and a circumscribed polygon are two axiomatically different species, and an unbridgeable discontinuity is generated between them. Such singularities can only be

understood from a higher ordering principle, and may not even be understood as an anomaly by the investigator.

(2) There are two cases where the thread reaches a limit of extension with respect to the **{EVOLUTE}** ***and to the*** **{INVOLUTE},** ***that is, one is a minimum, the other a maximum; and, both relate to the extension of the*** **{OSCULATING RADIUS}.** ***It is obvious that when the thread reaches the cusp, at one end of the*** **{EVOLUTE},** ***the curvature of the*** **{INVOLUTE}** ***is minimal; and from there, the curvature of the*** **{INVOLUTE}** ***cannot but increase as the radius of osculation is getting shorter. When the thread is the shortest, at the other end of the*** **{EVOLUTE},** ***the curvature of the*** **{INVOLUTE}** ***reaches its maximum. In his treatise on the*** **{HOROLOGIUM OSCILLATORIUM},** ***(Part III, proposition X), Huygens has shown the beautiful harmonic proportionality of such a minimum/maximum relationship between an involute and its evolute.***

(3) We have shown how the measuring principle of the curvature of the Earth was conceived by Leibniz, in {**LEIBNIZ AND THE TRANSCENDENTAL PROCESS OF OSCULATION} (97464/PB_001).**

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PART II

THE PRINCIPLE OF CONSTANT AND MOST RAPID DESCENT: THE ISOCHRONE AND THE BRACHISTOCHRONE CURVES.

During the month of March of 1686, Leibniz launched a powerful polemic against Descartes' false conception of "physical force", and of what he claimed was a general law of movements. In his* {BRIEF DEMONSTRATION}*…, Leibniz demonstrated the error of the Cartesians, by comparing a falling* {ONE POUND BODY A}*, released at the height of four yards, with* {A FOUR POUND BODY B}*, released at the height of one yard. The proportionality that Leibniz used in his little experiment satisfied everyone's "perception", and especially Descartes, to the effect that the values of the two bodies A and B appeared to be the same, and that since the same force was required to raise both of them to their respective heights, therefore, in both cases, the force of fall "must have been the same." The reader should consult the delightful dialogue that Philip Valenti has written on this subject, in* {LEIBNIZ AND DYNAMICS: A DIALOGUE}*- 98025PHI001-* {PEDAGOGICAL EXERCISE} *of Saturday Briefing, March 28, 1998.

It turns out that Descartes was totally wrong, and made a huge blunder in claiming that both objects had the same force, and that the force of each object, A or B, was equal to its mass multiplied by its speed. Leibniz, on the other hand, showed not only that, in certain cases, such a force may appear to correspond to the mass of the object multiplied by the square of the velocity, but also it turns out to be totally false. Indeed, the notion of* {FORCE} *that Leibniz was using implied much more than the simple proportionality of mass and velocity of falling bodies.

The great dispute that ensued on this question of the* {LIVING FORCE (VIS VIVA)} *led Leibniz to create an entirely new problem, out of thin air, that is, the problem of the* {ISOCHRONE CURVE}*, which represents one of the cornerstones of his calculus, and of his* {LEAST ACTION PRINCIPLE}*. As Leibniz wrote, in a letter to L'Hospital, on Jan. 15, 1696: "I have demonstrated that force must not be conceived as a composite of speed and magnitude, but by its future effect." In other words, as Valente showed in his dialogue: if you have a body weighing 1,000 pounds which hits you at a speed of 1/100th miles an hour, you will not be subjected to the same "effect", that is, you will not be hit by the same "force", as that of a body weighing 1/100th of a pound, and which strikes you at 1,000 miles an hour! Thus, Leibniz designed his experiment, on the issue of* {FORCE}*, in such a way as to enable people to make discoveries based on* {POTENTIAL FORCE}*, on the* {FUTURE EFFECT}*, on* {FINAL CAUSALITY}*; that is, based on the idea of* {PURPOSE} *and* {INTENTION}*. To wit: given the property of increased acceleration of a body in a free fall, 1) find the* {ISOCHRONE}*, that is,* {THE CURVE OF CONSTANT TIME OF DESCENT}*, and 2) find, among an infinity of such possible curves, the one which is the* {MOST RAPID ISOCHRONE}*. The way that Leibniz formulated the problem was as follows*: {"FIND A CURVE ALONG WHICH A HEAVY BODY ROLLS CONSTANTLY DOWNWARD TOWARDS THE HORIZON WITHOUT ACCELERATION."} *The Cartesians, and especially, Abbot Catelan, were totally baffled by this problem; however, both Christian Huygens and Jacques Bernoulli did make the discovery of the curve separately, and on their own.

Pedagogically speaking, the problem that the discovery of this curve poses is actually very beautiful, because it is a good example of the Leibnizian method of generating a **{PLATONIC IDEA}** *with respect to Physics:* **{DISCOVER THE PURPOSE OF A PROBLEM BEFORE YOU DISCOVER THE SOLUTION TO THE PROBLEM}.** *Think, for a moment, that even before knowing whether an* **{ISOCHRONE CURVE}** *really exists, or not, in nature, your mind has to establish two things: one, you must admit that, since the velocity of a heavy body falling freely in the vertical does accelerate, there must exist a hypothetical pathway along which that same body should be able to travel downward at a constant speed, that is, without any acceleration whatsoever; and two, for that same* **{REASON},** *there should also exist some unique pathway along which that same falling body must acquire, and maintain, a constant maximum velocity. It is this problem of the* **{ISOCHRONE CURVE} and of the {ISOCHRONE-PARACENTRIC CURVE}** *which convinced the Bernoulli brothers to become the first, and most ardent followers of the Leibniz calculus. Inspired by the new method of Leibniz, the Bernoulli brothers returned the favor by challenging Leibniz with the construction of the catenary and of the brachistochrone, the solution of both required that they master the integral and the differential calculus.*

In the same spirit of discovery, seven years later, in 1696, John Bernoulli announced his discovery of the **{CURVE OF MOST RAPID TIME}.** *And, this is how Leibniz immediately reacted to it in a letter: "Finally I come to your Problem: find the curve that one might call* **{TACHYSTOPTOTE}***; that is, of the most rapid descent. This problem is absolutely marvelous, and because I could not take my eyes off of it, its beauty attracted me to it like the apple attracted Eve." (C.I Gerhardt, M.S. III p.288.) Leibniz was ecstatic, not only because the problem itself was very elegant, but also, because Bernoulli had shown that he had finally grasped the purpose of the method of the Calculus. Bernoulli named his discovery, the* **{BRACHISTOCHRONE}.**

The following two Leibniz translations: **{ON THE ISOCHRONE CURVE, ALONG WHICH A HEAVY BODY DESCENDS WITHOUT ANY ACCELERATION},...(1689) and {NEW MODE OF APPLICATION OF THE DIFFERENTIAL CALCULUS TO DIFFERENT POSSIBLE CONSTRUCTIONS OF A CURVE FROM A PROPERTY OF ITS TANGENTS}, (1694),** *represent a continued effort, on the part of Leibniz, to elucidate a* **{GENERAL THEORY OF NON-LINEAR FUNCTIONS},** *especially with respect to his* **{METHOD OF INVERSION OF TANGENTS},** *that he began to elaborate in 1671, but that he published in* **{NEW REFLECTIONS ON THE NATURE OF CONTACT ANGLE AND OF OSCULATION}...,(1686); {ON OPTICAL CURVES AND OTHER QUESTIONS}, (1689); {CONSTRUCTION OF A CURVE FROM AN INFINITE NUMBER OF LINES WHICH ARE ALL WELL ORDERED, CONCURRENT AND TANGENT TO IT},...(1692); and {GENERAL CONSIDERATIONS ON THE NATURE OF CURVES, CONTACT ANGLES AND OSCULATION}...,(1692).**

Even though Leibniz chose to write up these ideas quite late in his life, he had already developed them, quite early on, during his first sojourn in Paris, in 1671, at the age of 26. Without any formation in mathematics whatsoever, the young Leibniz learned the basics of geometry from Fermat, Pascal, Desargues, and Huygens, but he mostly learned through the art of discovering.

The last piece translated in this series: **{COMMUNICATION ON THE SOLUTION TO THE PROBLEM OF THE CURVE OF MOST RAPID DESCENT PROPOSED TO GEOMETERS BY Mr. JOHN BERNOULLI}..., Acta Eruditorum, May 1697,** *is an historical account of the discovery of the cycloid before Bernoulli made the discovery of the astonishing property of the* **{BRACHISTOCHRONE}** *curve. Although, Leibniz gave a brief account of his own ideas on the ordinary cycloid, it is important to mention that he was aiming at drawing attention to the question of the* **{METHOD OF DISCOVERY},** *and at socializing how a discovery is self-generating; that is, give geometers the opportunity of discovering by themselves the* **{PURPOSE AND PROPERTY OF A CURVE}** *which is not given, as opposed to finding what can be derived from a given curve, once it has been discovered. From this vantage point, the reader should pay close attention to the fact that the discovery of the method underlying the Leibnizian calculus cannot be made without walking through it, step by step. In point of fact, the discovery of the method of changing axioms implies that you make use of it; and that, in doing so, you discover the pathway of increased density of singularities by walking through it. The case of John Bernoulli demonstrates this quite appropriately.*

Thus, Bernoulli discovered that **{THE PATHWAY THAT A RAY OF LIGHT TAKES, WHEN IT GOES THROUGH A MEDIUM WHICH INCREASES ITS DENSITY CONTINUALLY, IS THE LEAST ACTION PATHWAY THAT CORRESPONDS, NOT ONLY TO THE CURVATURE OF THE ORDINARY CYCLOID, BUT ALSO, MORE DRAMATICALLY, TO THE CURVATURE OF THE MENTAL PROCESS OF DISCOVERY ITSELF; THAT IS, THE CURVATURE OF CHANGE GOING THROUGH A HIGHER DENSITY OF ASSIMILATION OF VALID HYPOTHESIS.}**

Bernoulli was able to solve two different, but very much interconnected problems, with a single and very powerful discovery. On the one hand, he established a higher hypothesis for the Tautochronic cycloid of Huygens, and for the Fermat hypothesis of the **{LEAST TIME PATHWAY}** *of light going though media of different densities; and he did it, by providing the scientific world, for the first time in history, with the Leibnizian idea of a specific curve that expressed the general property of a maximum; that is, of optimizing in the sense of maximizing all efforts to reach the* **{CURVATURE OF TOTAL OPTIMISM},** *confirming the Leibnizian optimism of the best of all possible worlds. In addition, Bernoulli's discovery destroyed, in one very simple and beautiful experiment, the Newtonian and Cartesian fatalistic fallacy of the so-called objective world dominated by the push me, pull me, mechanical action of gravity, and asserted the fundamental, non-entropic, and subjective character of science, as exemplified by his assimilation into a* **{HIGHER NON-ENTROPIC INTEGRATION}** *of both the phenomena of mechanics, and the phenomena of optics, with those of the curvature of mental discovery itself.*

This type of approach should shed some light on the kind of questions raised by Lyndon H. LaRouche concerning the projection of photons, the question of refraction of light in a vacuum, and the self-propagation quality of radiation more generally. This sort of self-reflexivity common to all hyper-geometric work developed by Leibniz and Bernoulli, and later, by Carl Gauss and Bernhard Riemann, indicates the general path of the never reached, but always perfected pathway of paradoxical inversions, from the **{NON-LINEAR CHANGE OF CURVATURE} to**

the {NON-LINEAR CURVATURE OF CHANGE.} ***Thus, the cycloid-brachistochrone represents but an approximation of the sought for*** **{CURVATURE OF CHANGE};** ***a still closer approximation will be achieved by applying the same method to the integral of the cycloidal and elliptical multiply connected cyclical action of the catenary modular function*****.**

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ON THE SUBJECT OF THE ISOCHRONE CURVE, ALONG WHICH A HEAVY BODY DESCENDS WITHOUT ANY ACCELERATION, AND ABOUT THE CONTROVERSY WITH ABBOT D. C., by G.W. Leibniz, Acta Eruditorum, Leipzig, April 1689.

"Following the publication in the Acta of March 1686, of a demonstration against the {Cartesians} where I have revealed what the true nature of the forces was, in demonstrating that what is conserved is not the quantity of motion, but the amount of potential which is not to be confused with it. But, a French scientist, Mr. Abbot D. C. (Abbot Catelan) (1) took their defense without having understood well enough the meaning of my arguments. He believed, in fact, that I was putting into question certain other accepted principles. He passes them in review in the News of the Republic of Letters of June 1687, p.579, where he declares that he fails to see the contradiction that I thought I had discovered. In fact, it has never come to my mind that I would put these principles into question; and this is what I took the trouble of explaining to him in the News of the Republic of Letters, of September 1687. He went on a digression, which was completely superficial with respect to what was the real issue of the controversy, simply in order to avoid responding to my objections. In fact, as long as the heights remain the same for all falling heavy bodies, they will acquire or lose the same force, independent of the time of the descent, more or less, depending on the inclination.

"In order to assure that our debate will contribute to the progress of knowledge, and taking the opportunity to show that the time factor has no role to play whatsoever in this case, and neither does the distinction between isochronic and anisochronic potentials, I have submitted to him in September 1687, in these News, the following problem, which I was in the process of solving, as I was enunciating its very formulation; which, as everyone can see, is not without elegance: {find the isochrone curve along which a heavy body will fall uniformly, that is to say, in equal time periods, while also approaching the horizon, and will carried downward without acceleration, at a constant speed}.

" But, Mr. Abbot D. C. did not go any further, either because he gave up on the problem, or because he discovered that I was right and he finally adopted my point of view. In contrast, in the News of the Republic of Letters of October 1687, the celebrated {Christian Huygens} considered this problem worthy of his attention, and has himself given a solution, which is in total agreement with mine; but he did not provide a demonstration, and he did not explain how to choose between the curves of the same type, that is the term he used, as he says, which are appropriate. (2) So, I have decided to complete this, right now, which I would have done earlier, if I had not waited for the results of the work from Monsieur the Abbot.

"{Problem}: Find the plane curve along which a heavy body falls without acceleration.

"{Solution}: It is the squared-cubed paraboloid βNe (that is to say, where the product of the square of the base NM with the parameter aP is equal to the cube of the height βM), whose tangent βM at the summit β is perpendicular to the horizon.

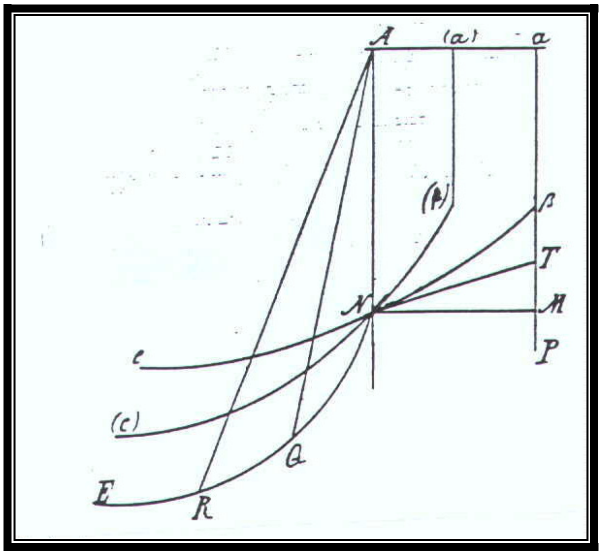

Figure 5. The isochrone curve: the curve of constant time of descent.

"From any point N of this curve, consider a heavy body, which has already acquired a certain speed downward after it has fallen from level Aa, located higher than β at a height aβ, which is equal to $4/9^{th}$ of the parameter of the curve. In such circumstances, this body will continue its fall, as we wished, uniformly along the curve Ne that we can extend freely.

"{Demonstration}: Take NT as the tangent line to line βNe at N, cutting βM at T. In all cases, (according to a well known property of tangents to this curve), TM will be to NM as the square root of aβ will be to βM. (That is, TM/NM = square root of aβ/βM. P.B.) Consequently, TM will be to TN as the square root of aβ over aβ + βM, that is of aβ to aM. So TM is to TN as the vertical speed that the body has acquired at N, {when it follows the curve} (that is to say the speed at which it continues to have as it approaches the horizon) to the vertical speed that it would have at N, if it was no longer falling along the curve but, if it were possible, {freely} (this according to the properties of an inclined movement). But, this speed of {free fall} is itself, with a determined constant, equal to the square root of aM to aβ; in fact (as we can see with the movement of heavy bodies), the speeds in free fall correspond to the square roots of the heights aM (where these falls begin in order to produce these velocities), and consequently, the vertical speed of a body at any point N, is of a constant speed when it is equal to the product of the square root of aβ to aM, and of aM to aβ, that is a ratio of equality. This vertical speed along the curve is therefore itself constant, that is to say, identical at every point of this curve Ne, which is what we wanted to achieve.

"{Conclusion}: 1) A heavy body which has acquired a certain speed while falling from the height of Aa, for example, could come down along an infinity of isochronic curves all of

the same species, from the same point N. By this, I mean that, different only by the value of their parameters, the curves Ne, N (e), NE, are all square-cubic types of paraboloids, and therefore all similar curves. Moreover, all paraboloids of this type are acceptable here, under the sole condition that they be situated in such a way that the distance aβ or (a)(b), between the summit and the level a (a), where the fall begins, be equivalent to 4/9th of parameter be or (b)(e) of the curve. In order to go from a (a) to N, before going down the curve at a constant speed, it is not important that the body had traveled the trajectory (a)(b) N, or any other trajectory, nor even that it might have acquired that same speed, and that same direction, by any other means than from a fall. (3) However, from this infinity of isochronic curves, along which a body can pursue its fall from point N without any acceleration, the one which would have the most rapid descent is the one for which N is the summit, that is NE, whose tangent is the straight line AN, the perpendicular to the horizon.

"2) The time of descent on the straight line aβis with the time of descent along curve βN, as a ratio of aβ is to half of the height βM ; consequently, if βM is the double of aβ, the times of descent of aβ and βN will be the same. The reason for this should be obvious: the times of a uniform descent are between themselves relative to their heights, and as Galileo has demonstrated, the time that a body takes to travel through an accelerated movement of height aβ is double that of the one which travels a height equal to βM through a constant movement (as it is the case here along the curve βN), that is, at the speed that it has acquired at point β of the accelerated movement.

"I must confess that I have not raised this problem at the intention of first-tier Geometers, who are well trained in what I would call superior analysis, but, on the contrary, at the intention of those who share the beliefs of this {French savant} who appeared shocked at my accusations against the {Cartesians} of today (who paraphrase their master rather than imitating him). In reality, aside from the fact that these men attribute too much credit to their fashionable precepts, such men are making too much fuss over their analysis, which they regurgitate among themselves, to the point that they end up believing that, thanks to it, they are capable of overcoming everything in mathematics (naturally, as long as they give themselves the trouble of making a few calculations); and this is very much to the detriment of the sciences, that these researchers, too confident in their old inventions, have become too lazy to make progress. So, I wanted to provide them with some material for their Analysis of a problem that does not require so much extended calculations, but rather some finesse.

"If one of them were to complain that I have pulled the rug from under his feet, he could always look for {another isochrone}, which is a cousin of the former, and along which a body is moving away (or moving forward) constantly (uniformly), not as we have supposed above, toward a horizontal line, but toward or from a determined point. The problem would therefore be: {Find a curve along which the fall of a heavy body would move away, or would move ahead, uniformly, from, or toward, a given point}. (4)

"Such a curve would be NQR if, once we have traced a number of straight lines AN, AQ, AR from the fixed point A, toward the curve, the portion of AR which exceeds AQ were to be with the portion of AQ which exceeds AN in a ratio of time of the fall along the arc QR with respect to the time of the fall along NQ."

NOTES:

(1) In the Latin original, C.I. Gerhardt identified A.C. improperly as* {ABBATE DE CONTI}*. However, the French translator, Marc Parmentier, properly identified the initials D.C., that Leibniz used in his manuscript, to stand for Abbot Catelan, who was an obscure and insignificant zealot of Descartes.

(2)Christian Huygens,* {THE PENDULUM CLOCK OR GEOMETRICAL DEMONSTRATIONS CONCERNING THE MOTION OF PENDULA AS APPLIED TO CLOCKS}*, The Iowa University Press, 1986. In Part III, Proposition 9, P.84, Huygens writes: "Find a straight line equal to a given portion of the curve of a paraboloid; namely a paraboloid in which the squares of the perpendicular to the axis are related to each other as the cubes of the abcissas are to the vertex."

(3) Here, Leibniz is setting the conditions whereby one can discover that a curve of most rapid descent might have its acceleration caused by some other law than the so-called "law of gravity." This is the same condition that John Bernoulli will advocate in his discovery of the Brachistochrone problem, as it applies to the least time trajectory of light in a body of increasing density. Indeed, as Lyndon H. LaRouche Jr. has shown, gravitation does not depend on the so-called Newtonian attraction of heavy bodies at a distance, but is rather derived from the isochronic curvature of physical-space-time.

(4) Note how Leibniz creates, here, a brand new problem rather than a new curve. What he emphasizes, again, is the discovery of the* {PROPERTY OF THE CURVE}*, rather than the nature of the curve itself.

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6

COMMUNICATION ON THE SOLUTION TO THE PROBLEM OF THE CURVE OF MOST RAPID DESCENT PROPOSED TO GEOMETERS BY MR. JOHN BERNOULLI, AND OF THE SOLUTIONS THAT BOTH, HE AND MR. LE MARQUIS DE L'HOPITAL, HAVE ASKED ME TO PUBLISH, INCLUDING THE SOLUTION OF ANOTHER PROBLEM THAT MR. BERNOULLI HAS LATER PROPOSED, by G. W. Leibniz, Acta Eruditorum, Leipzig, May, 1697.

"The art of submitting problems to geometers is a generalized practice, and is profitable for everybody, providing it is not done with the intention of bragging about one own successes, but is done, on the contrary, with the idea of inciting others to discover; that is, in such a way that discovery is enriched with the particular method where each personality contributes to the art of invention. It often happens that men of science, knowledgeable in the accepted practice of Analysis, give too much credit to the methods that they have learned, and, without looking beyond that narrow horizon, end up being satisfied with the commonly accepted theories, and this, to the detriment of science itself. This happens when one is persuaded that no problem is beyond one's ability to resolve, and he no longer looks for something new, but falls pray to laziness and vanity. There exist no better remedy, to get them out of their lethargy, than to submit them to problems, which distinguish themselves by their elegance, as well as their usefulness, especially when they demand of them more subtlety than labor.

"Therein lies, in my sense, the reason for the success of the method of infinitesimals that I have initiated with respect to differences and summations (and which became known as the differential calculus), and of its adoption by a number of imminent individuals: it turned out to be the most appropriate method for solving outstanding problems. Indeed, I began to validate that method when, in response to {M. Abbot D. C.} in the News of the Republic of Letters, where he had raised some objections to my work on {dynamics}, and thus lending too much credibility to the Cartesian methods, I got the idea of responding to him, as well as to anyone who had the same sentiment, by showing that I had solved the relatively easy problem of the {isochronic curve} according to which a falling heavy body would approach the horizon at a constant speed. But, the Cartesians responded with a resounding silence, and it was {M. Huygens}, who, after considering the elegance of the problem, provided the solution. And, since he had deliberately left out other considerations, I completed them myself by providing a demonstration in the {Acta Eruditorum}; which I did more with the idea of putting an end to the debate, than to gain some advantage.

"But, since there is always a certain continuity in everything, my demonstration had the effect of suddenly inspiring {Jacques Bernoulli}, who, up until that time, only had an occasional flirting with the differential calculus that I had published in the Acta, and without getting anything out of it. But, since he grasped the importance of this method for questions of mathematical-physics, he then submitted to me the problem of the Catenary Curve that Galileo had tackled without any success. Having solved the problem, I could have published

my solution and thus enjoyed all by myself the praise of fame, without sharing it with anyone; but I chose, on the contrary, to invite others in concert with me, to participate and support me in establishing this very beautiful method. Indeed, it is certain that great minds are often attracted by glory, and will more likely endeavor more from what they prefer to develop themselves, rather than to take ready made solutions from others. So, I let it be known that I had the solution of the problem that Galileo searched in vain, but, that I had also made the decision to wait a year before publishing it, in order that others might have sufficient time to, either put together their own method, or to ponder sufficiently on mine, so they could make use of it with full knowledge. The challenge was a total success.

"Developing his own method, {M. Huygens} (who we mourn the loss of today) arrived at an incomplete solution (as he willfully admitted later). Next, it was the study of my calculus that led {M. John Bernoulli} to the right answer, after he had made the connection with the area of the hyperbola, as I had done myself, but with the only difference that he found the construction by means of the rectification of a parabolic curve, while I made use of Logarithms. This resounding success provided the Bernoulli brothers with a wonderful opportunity which enabled them to later accomplish marvels with this calculus, so much so that, from now on, this method is as much theirs, as it is mine. A little bit later, Huygens, who up until then had given it little interest, personally made the experiment of its merits, and let it be known; others, and particularly, {M. le marquis de l'Hopital}, in France, and {M. Craig} in England, fell into steps with them. But, first and foremost, {M. Jacques Bernoulli}, professor, made on the Curve of the Sail, and on the Elastic, the most outstanding results from Bale. Meanwhile, {M. le marquis de l'Hopital} has recently produced a very beautiful work on the principles of the method, remarkably well illustrated with numerous and refined examples.

"Finally, a short while ago, {M. Bernoulli}, professor at Groninguen, has taken on the study of another problem, that of {the trajectory of the most rapid descent}, a problem that {Galileo} had also attempted to solve, but without success; a problem whose beauty and applications have nothing to envy over those of the {catenary}. He solved it, and he invited others to do the same. This shows how two illustrious problems that Galileo had identified incorrectly, and had attempted in vain to solve, have found their actual solution in the method of our calculus.

"Naturally, the genius and shrewdness of {Galileo} is in no way put into question, because, during his time, the art of analysis was not sufficiently advanced, and its superior part, the analysis of infinitesimals, was still in the dark; so, he could have hardly been able to discover solutions of this type. He had mistakenly taken the {catenary curve} for a Parabola, and considered the {trajectory of most rapid descent} to be the Circle; which is quite far from the truth, since the determination of the {catenary} is acquired by Logarithms, that is to say, by means of rectification of parabolic arcs, and the Trajectory of a most rapid descent is gotten by the rectification of circular arcs.

"However, {M. Bernoulli} approached the whole question under much better auspices, not only was he the first to discover that the curve of most rapid descent was the Cycloid, but he realized that this Brachistochronic curve was also the bearer of another secret: that is, the

curve formed by the light of rays which are modified in a medium of constant change. Huygens had come across that question in his Treatise on Light, but without attempting to resolve it.

"By way of the Acta of Leipzig, {M. Bernoulli} has thus launched a public invitation to all geometers, challenging them to find a solution to this problem within a delay of six months, and has asked me, in a letter, to spend some time on this. I could have acted as if the numerous other tasks that overwhelm me could excuse me from this, and decided to push off this new task, but, the beauty of the problem attracted me to it in spite of myself, and I succumbed to the ascendancy of its charm.

"Soon, I had the opportunity to see my dream realized. So, I communicated my solution to the Author of the problem, and as soon as he saw that we were in agreement, he transmitted his solution to me for safe deposit, with the agreement that I would publish it at the appropriate time. But the six months deadline passed, and nobody else, but us two, had a declared solution to offer.

"{M. John Bernoulli} could have made his solution known and laid claim, practically for himself alone, to the glory of a very elegant discovery; I would have encouraged it, myself, if we had preferred to work for our own personal glories rather than for the general interest. However, after deliberations, we came to the conclusion that, for the progress of science, and in order that the problem remained graven on memories, other people had to participate in this success, and we decided, from a common accord, to prolong the delay for another six months, even though we knew roughly ahead of time, as I predicted to him in a letter, who were the ones that would be able to discover the solution, providing they made the effort of using my already published discoveries.

"So, now we can establish that they have, presently, answered our call. It is very interesting to note that only those who we thought could accomplish this task, have, indeed, resolved the problem, that is to say, exclusively those who have clearly understood the secrets of my differential calculus. Aside from the brother of its Author, I forecast, in France, {M. le marquis de l'Hopital}, and I added, among others, {M. Huygens}, if he were still alive, {M. Hudde}, if he had not abandoned such work, and {M. Newton}, if he had agreed to work on it at all. All would have been up to the task. I am recalling all of this because I do not want people to think that I hold in contempt those distinguished men who did not have the possibility, nor the leisure of showing us some interest.

"The solution of John Bernoulli was sent to me last year, in the month of August. If you want to know the results from M. Jacques Bernoulli, and their dates of publication, all you need to do is consult the Article he has sent directly to the Acta. As for {M. le marquis de l'Hopital}, he sent me his solution in a letter, last March. In addition to his solution, M. John Bernoulli elaborated an original method which brought him to his results, and since he had made use of two approaches, here we show only the indirect one, because it stems from dioptric considerations, which doesn't diminish its great elegance in any way. He has another method which is more direct, and which stems straight from the heart of the problem; and I know he will agree to show it to anyone who asks him for it.

"By posing these types of problems of Maxima and Minima in this form, this method reflects something totally new which goes way beyond the ordinary questions of Maxima and Minima. It was on these questions that {Fermat} became the first to establish his method, and that later, {Descartes, Hudde} and {Sluse} among others, will adjust their own well-known approaches. Indeed, in terms of their own choice of subjects, everything is practically reducible to finding the maximum or minimum ordinates of a given curve, which is nothing but a corollary to the ordinary Method of tangents, that is to say of the direct method. But, in this specific case, what we are looking for, is the actual curve itself, which has to satisfy {optimally} a given condition, and its nature is often so obscure that the data do not even manifest any property of its tangents, and consequently, it is not easy to bring the problem back, even to the superior method of the inverse of the tangents. This would also be the case for the problem of the {catenary}, unless certain preliminary conditions were first discovered. (1) In fact we must first discover which is the form of the curve, of a given length between two determined points, such that its center of gravity would be the lowest possible. This shows how much, up to now, the method of Analysis is far from perfection, no matter what extreme certain people went to, in order to promote their own Methods. (2)

"But, the solution of John Bernoulli has another precious and supplementary advantage: we now hold the solution to two problems of Dioptrics which are of great consequences, and whose apparent difficulty had dissuaded Huygens, as well as everybody else after him, from tackling this problem. We are now able to determine the continuous curvature of light rays, as well as the curve that describes their reflections (the brachistochrone).

"I must not indulge in explaining my own solution, here, because it is similar to that of the others; satisfied of having determined the sought for curve, I did not have any time to add any details. The only point worth mentioning, however, is perhaps the following: as my calculus has shown, the sought for curve is a figure representing circular segments. In point of fact, find the curve ABK generated by the circle which goes through its lowest point K, and is tangent to point G on the horizontal line going through A.

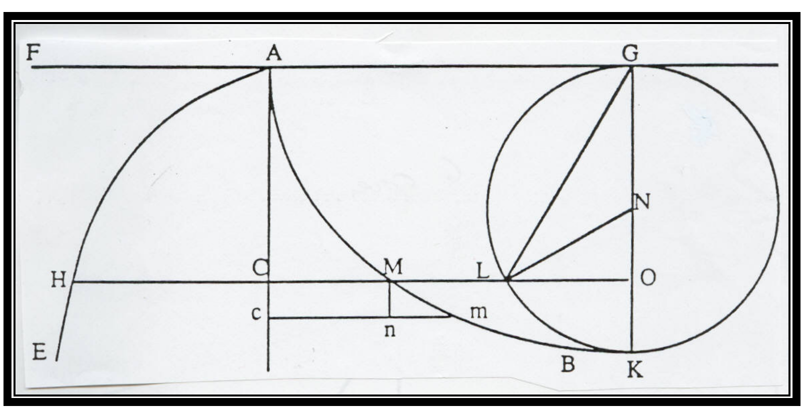

Figure 6. The ordinary cycloid: AB is the curve of most rapid descent.

“Trace perpendicular lines to the vertical axis AC such that they cut it in C, cut the curve in M, cut the circle in L, and cut in O its vertical diameter GK. Suppose that the ordinates CM are proportional to the circular segments, and that the product of the half-radius of the circle by (and) CM is equal to the segment between the arc of the circle and the chord GL. Under these conditions, the portion AB of the curve which lies between the two given points A and B, will be the pathway along which a body falling of its own weight, will travel with the maximum possible speed, from point A to point B. We can also show very easily, that this arrangement of segments is identical with the ordinary Cycloid: OC being equal to the half of the circumference GLK, and LM being equal to arc KL, we will get: OL + CM is equal to GL. Take the center of the circle N, and join points N and L, we can see that the product of the half-radius by (and) OL + CM is equal to triangle GNL.

“{M. John Bernoulli} just proposed another problem of great interest and purely geometric: {find the curve which has with an arbitrary straight line going though a fixed point, two intersection points such that the segments between the fixed point and the two points on the curve, when elevated to a certain power, has a sum equal to a constant}. I would like to propose a solution. I have been told that this solution is precisely the one which the Author of the problem had himself conceived, and which he preferred, while we also had at our disposal many other ways of solving it. Our solution is as follows: on Figure 7., we are seeking the curve CEFD whose points E and F intersecting with an arbitrary straight line going through a fixed point A are such that: AE>e + AF>e = constant b

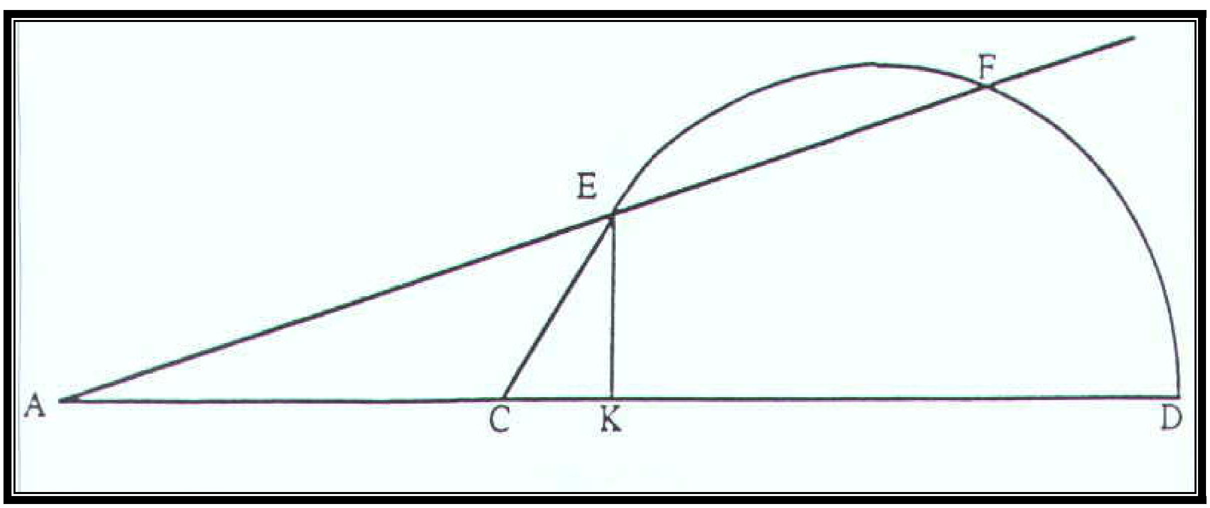

Figure 7. Secant AEF cutting a circle.

“We note AE (or AF) x, and EK y, let us take a straight line for unity and a constant C, we will have: C/y = bx>e-1 – x>2e-1, an equation which will give you the nature of the curve you are seeking, as well as a means to determine points CEFD.“

NOTES:

***(1) The problem here involves the paradox of discovery itself; that is, the act of discovering a curve which is* {UNKNOWN AND NOT GIVEN TO YOU} *from the start, but which is being discovered as you append tangents to it. In other words, you must start with the idea of a* {CERTAIN PHYSICAL PROPERTY}, *from which a curve is generated with the use of tangents, not with the idea of attaching tangents to a curve after it has been discovered. This is what Leibniz calls the method of the* {INVERSION OF TANGENTS}, *where you are seeking to discover a non-existent* {EVOLUTE} *from the concurrence of perpendiculars elevated from a given {*INVOLUTE}*; which is the kind of problem that arises with the generation of envelopes, the method of discovering the One of the Many, the* {EVOLUTE} *of an infinity of* {INVOLUTES}.**

***(2) There can be found a most simple, and beautiful construction for the catenary/tractrix curves, which is generated by the Leibnizian method of inversion of tangents, and which is presented in* {APPENDIX 2, P.104}. *If anyone wants to take on this Leibniz challenge, the sought for construction can be derived from this simple Leibnizian proposition:* {GIVEN A FAMILY OF CIRCLES CUTTING A CURVE AT RIGHT ANGLE, FIND THAT CURVE. G.W. Leibniz, SUPPLEMENT TO GEOMETRIC MEASURES}...{ACTA ERUDITORUM*}*, M.S. V. V p.301.**

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PART III

THE PRINCIPLE OF THE TRACTRIX AND THE METHOD OF DISCOVERY BY INVERSION OF TANGENTS

Leibniz is not only the first geometer to have scientifically determined the properties for the generation of the **{CATENARY CURVE} (See FIDELIO, Spring 2001),** ***he is also the first to have rigorously defined the physical-geometric properties for the generation of the*** **{TRACTRIX CURVE}.** ***The following translation from {*****ACTA ERUDITORUM},** ***September 1693, establishes the record of this latter discovery, and gives an exciting account of its pertinence for higher integrations of physical processes.***

The crucial feature of this discovery does not reside in **the {TRACTRIX CURVE}** ***itself, but in the*** **{GENERAL PROPERTY}** ***of generating the curvature of such a curve, and the physical requirements for the determination of its complex motions. Thus, Leibniz established the*** **{TRACTION MOTION}** ***as a*** **{COMPOSED MOTION}** ***made up of a*** **{UNIFORM MOTION},** ***and of a*** **{RETARDED MOTION},** ***both of which are ordered by the same multiply connected least action principle. Also, this*** **{COMPOSED TRACTION MOTION}** ***has the singular property of*** **{NATURALLY EXECUTING THE COMPLETE QUADRATURE OF THE CURVE};** ***that is, it accounts for the complete, and orderly, covering, or integration, of the entire area under the curve. The elegance and simplicity of such a construction is, for Leibniz, the token of its scientific fruitfulness. It is both geometrically and epistemologically sound with respect to the best, or the optimum displacement of a multiply connected motion.***

Huygens considered **{TRACTION}** ***as the simplest and most perfect method for this type of construction. After explaining briefly the mechanism of the traction motion, that is, by " pulling a point, which by its own weight, or by some other means, would offer some resistance, being attached at the end of a thread, or from an inflexible rod, which we would simply pull forward from the other extremity," Huygens concluded: "If this description can be considered geometrically exact, by mechanical standards, we should consider that, what we have here, with the quadrature of the hyperbola, is the perfect construction of all of the problems that can be reduced to this type of quadrature."***

Indeed, Leibniz was very happy that Huygens approved of his hypothesis, because he had already promoted, with this article of the Acta, the idea of generalizing the **{TRACTION MOTION}** ***for application into more complex domains. It is as if he were conceptualizing the whole thing for the purpose of developing a machine-tool design. Leibniz was considering some kind of*** **{INTEGRAL GENERATING MECHANISM},** ***some sort of*** **{INTEGRAPH}** ***that would permit him to construct higher and more*** **{COMPLEX QUADRATURES},** ***with the use of different methods, including that of*** **{TRACTION}.** ***This is why he responded to Huygen's remark by asserting:" As far as my general construction of quadratures by means of traction is concerned, it is sufficient for it to be theoretically exact in order for it to be scientific, even if it were not practicable in reality. Most geometrical constructions are of this nature, when they are initially elaborated."***

From this standpoint, it should be clear to anyone that the calculus of Leibniz is not, as it is professed falsely to be in academic circles, a method of generating equations for the purpose of measuring this or that; it is, quite to the contrary, a new method of geometric construction, an algorithm for measuring change in multiply connected physical space-time in the small. What is required to determine the measure of change in physical space-time is to define the rules of composition of the **{CHARACTERISTIC OSCULATION}** *in the small. In other words, the most important feature of this new calculus is the discovery of the* **{CHARACTERISTIC}** *form of action; that is to say, the measurement of non-linear differences within small intervals of successive changes in physical space-time which are generated from an integrating summation as a whole.*

With this higher principle, Leibniz added to his method of **{INVERSION OF TANGENTS}** *a new physical process of* **{TRACTION MOTION}** *to reflect new implications for higher integrations. By pushing for higher integrations, a method that he had been developing for his calculus since early 1672, Leibniz reached for the highest form of inversion: that of the reciprocity between* **{DERIVATION}** *and* **{INTEGRATION}.** *It is from that standpoint that this application of the method of* **{TRACTION}** *is to be taken into account at the very opening of his paper, when he addresses the three levels of magnitudes of his* **{GEOMETRY OF MEASUREMENTS}:** *the* **{RATIONAL},** *the* **{ALGEBRAIC},** *and the* **{TRANSCENDENTAL}.**

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EXTENSION OF GEOMETRIC MEASUREMENTS USING AN ABSOLUTELY UNIVERSAL METHOD OF REALIZING ALL QUADRATURES BY WAY OF MOTION: ACCOMPANIED BY DIFFERENT PROCEDURES OF CONSTRUCTION OF A CURVE FROM A GIVEN PROPERTY OF ITS TANGENTS, by G.W. Leibniz, Acta Eruditorum, Leipzig, September 1693.

"The measurement of curves, surfaces, and most volumes, just like the determination of the centers of gravity, all comes down to quadratures of plane figures; such is the starting point of {geometry of measurements}, which is by nature different from {geometry of determination}, which involves only lengths of straight lines and by their means determines unknown points from others that are given. As a rule, we may naturally reduce this geometry of determination to algebraic equations, whose unknown has a determined degree. However, geometry of measurements is not, by nature, reducible to algebra, even if it happens, sometimes, that it is reduced to algebraic magnitudes (when we have to deal with ordinary quadratures); similarly, the geometry of determination is not of the domain of arithmetic, even when it happens (in the case where quantities are commensurable) that it is reduced to numbers, that is rational quantities. We can derive from this {three types of magnitudes, rational, algebraic, and transcendental}. The {origin} of algebraic {irrationals} resides in the {ambiguity} of the problem, that is to say, its {multiplicity} (1); in fact, it would be impossible to regroup under one single calculus the different values or solutions of a problem, unless it is done by means of roots, but then, at the exception of certain particular cases, they cannot be reduced to rational magnitudes. On the other hand, the {origin of transcendental magnitudes is infinity}, so much so that the {analysis} which corresponds to the {geometry of transcendentals} (to which belongs measurements) is very precisely the {science of the infinite}. (2)

"Furthermore, when algebraic magnitudes are constructed, what is required are determined motions which do not need material curves but only straight edges, or, when material curves are considered, only their intersection points are taken into consideration; on the other hand, in order to construct transcendental magnitudes, we have used, so far, the application of curves with respect to straight lines, that is we have adjusted one to the other, as in the case of the construction of the cycloid (3), or with the unraveling of a thread, or a leaf, when they are wrapped around a curve or a surface. If you wished to trace geometrically (that is by a constant and regulated motion) the Archimedian spiral, or the Quadratrix of the Ancients, you could do it without any difficulty by adjusting a straight line to a curve, in such a way that the rectilinear motion would be regulated from the circular motion. And that is why, contrary to what {Descartes} has done, I will not exclude such curves from geometry, because the lines which are so described are exact, and they involve properties which are very useful, and are adapted to transcendental magnitudes.

"There exist, however, other means of constructing curves, which involve the addition of a physical component. Such would be the case when the solution of a problem of geometry

of determination were to be found by means of light rays (which could be often done with great profit), or if one were to proceed, as I have done with the quadrature of the Hyperbola, or for the construction of logarithms by making a uniform motion, and a retarded motion by a constant rubbing motion, or else, by means of a string, or a heavy chain which produce the Catenary, or the funicular curve (la chainette). As long as the mode of its construction is exact, it will belong to theoretical geometry: as long as it is practical and useful, it has a right of application in reality. Indeed, any motion executed according to determined hypothesis is as much of the domain of geometry as is a center of gravity. (4)

"But, there exist a new type of motion, which, I think, I have been the first to make use of in the constructions of geometric curves, and I will say in what circumstance; because it seems to me, better than any other type, to belong to pure geometry, and resembles the tracing of curves by means of threads originating from umbilical points or from focus curves. In fact, the only condition that is required for the point to trace the curve in the plane is that it needs to be attached to the extremity of a thread located in the same plane (or in an equivalent plane), and that it must be in motion at the same time as the motion of the other extremity, but by a motion which is a simple traction, without any lateral impulse, which would not really work with a thread because of its flexibility; because the point has to be pulled in the direction of the tension of the thread which drags it along, that is in the direction of the thread to the extent that there exist no obstacle along the pathway.

"However, since a material thread never has the absolute flexibility that geometry requires, it could easily drag an engraver's point, in other words, the point producing the trace (which is free in the plane), in such a way that the motion of the engraver's point would represent nothing else but a simple traction; but to this material obstacle we could easily oppose a material expedient such that, when the tracing point is pressing down slightly against the plane to which it belongs, it is bound to it; and, such an expedient can be represented by a weight added to the point, or tied to it, in such a way that, by this added heaviness, the point weighs on the horizontal plane where it is suppose to define the pathway which traces the curve. (5)

"In this way, if the resistance of the weight, which impedes the motion of the point, is always stronger than the small residue of stiffness which is subsisting in the thread, the thread would be that much more tense, and its motion that much more regular; thus, the added weight would help the point trace properly the curve by traction only, and with no lateral impulse, which is the only condition that had to be imposed for the required motion. Furthermore, it follows that such a motion is remarkably suited for transcendental geometry, because it directly involves tangents that indicate directions for the curves, in other words elementary magnitudes that are infinite in number, but with unassignable lengths, in other words infinitesimals.

"It was a long time ago, in Paris, that I first imagined such a construction. The notorious Parisian Doctor, {Claude Perrault}, remarkable for his knowledge of Mechanics and Architecture, at the same time well known for his edition of Vitruvius, and who became one of the eminent life time members of the Royal Academy of Sciences, submitted to me, as well as to a lot of other people, the following problem, which he was not able to solve, as he

honestly admitted: that is, find the curve BB (Figure. 8) which a heavy point traces in the horizontal plane, at point B, or some equivalent point, and which is attached at the extremity B of a thread, or of a small chain AB; and when, by guiding the other extremity A of the thread AB along a fixed straight line AA, the weight of B is being pulled in the horizontal plane, (or in another equivalent plane) where the straight line AA is already located along with the motion of the thread AB.

"In order to make the causality of this process more intelligible, he was using a watch B in a silver jewel-case that he had attached to a small chain whose other extremity he would pull along a straight edge AA which was fixed on a table. In this manner, the lowest part of the jewel-case (located in the middle of the bottom part) was describing on the table the curve BB. When I closely examined that curve, (It was in the period when I was studying tangents), I immediately realized that the key to solving the problem resided in the fact that the thread was constantly tangent to the curve, that is to say that any straight line 3A3B is tangent to curve BB at Point 3B.

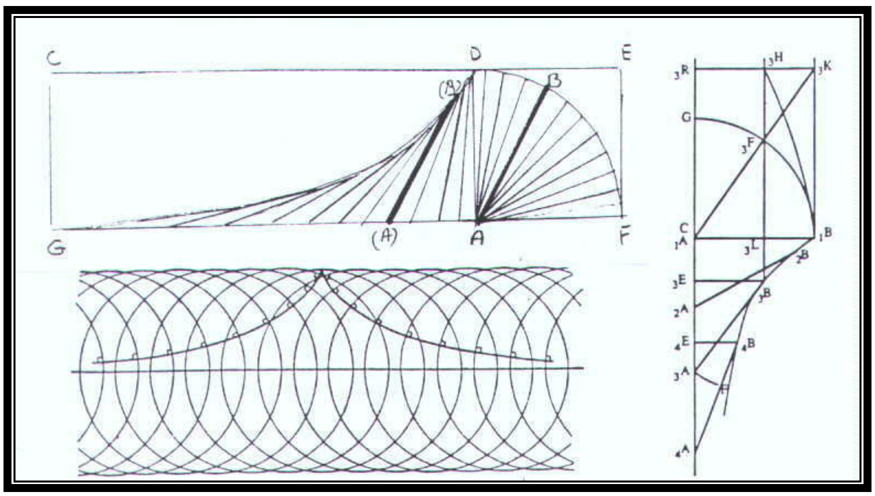

Figure 8. The tractrix curve.

"Here is a demonstration of this: draw an arbitrarily small circular arc 3AF, whose center is 3B, and whose radius is the thread 3A3B. Pull the thread 3BF from F, directly, that is to say, according to its own direction up to 4A, in such a way that from 3BF the thread goes to 4B4A; assuming that we proceeded for points 1B and 2B as we did for 3B, any point B would describe a polygon 1B2B3B etc., whose sides always fall on the thread; thus, by indefinitely diminishing arc A3F, and by finally making it vanish, we describe our tracing by the motion of continuous traction, where the lateral displacement of the thread is continuous but always {unassignable}, it is clear that the polygon is changed into a curve, which has this thread as its tangent. So, I realized by this, that the question could be reduced to a problem of conversion of tangents: find a curve BB such that the portion of its tangent between the axis AA and the curve BB is equal to a given constant. It was not difficult either to understand that the tracing of this curve could also be reduced to the quadrature of the hyperbola. (6)

“Let us draw, in fact, the circle 1BFG whose center is C, or A (where the thread 1A1B is at the same time the ordinate of the curve and its tangent), and whose radius is AB. This circle cuts the axis AE; given that 1BK is parallel to this axis, which the straight line CF cuts at K, 1BK will be tangent to the circular arc 1BF. Then, trace the straight line FLB going through F, and parallel to the axis AE, that line cuts 1A1B at L, and curve BB in B, from this straight line, trace LH equal to 1BK; by proceeding everywhere in the same manner, we get the curve of the tangent 1BHH, and we realize that rectangle 1B1AE is equal to the figure of the tangents, that is to the trilateral area 1BLH1B; for example the product of 1B1A and 1A3E will be equal to the triline 1B3L3H1B. Therefore, since we can find the area for the figure of the tangents by the quadrature of the hyperbola, that is to say, by logarithms, as everybody knows, it is also clear that we can equally get 1A3E and 3L3b, and consequently, any point 3B on the curve. Conversely, if we are given the curve BB, we will be able to construct the quadrature of the hyperbola, that is, the logarithms.

“I don’t want to explain all of this more extensively than what is necessary primarily because, in my opinion, the well known {Christian Huygens} has perfectly covered the subject; he told me, in a recent letter, that he just got an original idea for the quadrature of the hyperbola, which has been recently published in the History of the Works of Scientists. I can get a good idea of it thanks to the articles that the {Bernoulli} brothers have just published in the {Acta Eruditorum}, because, starting from the discoveries of Huygens, they have very judiciously applied a similar motion to describe the curve for which the portion of the tangent between the curve and the axis, that is BD (Figure. 9), is in constant relationship with a portion of the axis between a fixed point and the point of intersection of the tangent, that is AD, as is the constant proportion between two straight lines, N and M. This is what convinced me to finally publish my old reflections on this subject. (7)

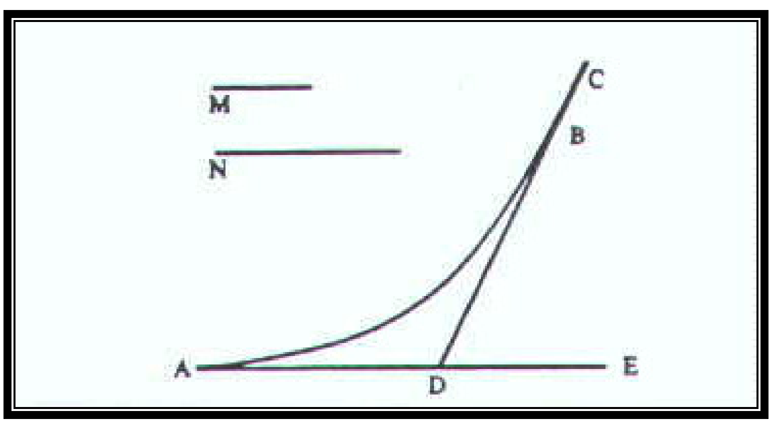

Figure 9. Ratio of the tangent to the axis.

“Right away, it was easy for me to understand that, once you grasped the relationship between the motion and the tangents, then you could use the same process to construct many other curves which would otherwise be more difficult to square. Because, even if you were to

suppose that AA is not a straight line but a curve, the thread would nonetheless be tangent to BB. Furthermore, even if the length of the thread AB were to increase or diminish while you are pulling on it, it would nonetheless remain tangent to it. That is why, whatever the relationship between CA and AB (Figure. 9) (for example if the AB's were the sinuses and the CA's were the corresponding tangents), several expedients could be used to regulate the motion of the thread so that it could move, while itself diminishing, according to a given law.

"This procedure gives us the ability to trace an infinity of solution curves representing a similar problem, for example, the curve which goes through a given point. If the point producing the tracing is being pulled simultaneously by several threads, you will generate several composite directions. (8) But, even if we were dealing with only one thread, we could vary its length by attaching to weight B, a wheel, or a rotating mechanism, such that it could describe a cycloid in the plane. We could also join to B a rigid straight edge which would be always perpendicular to the thread, or which would make a constant angle with it, or an angle which would vary according to some determined law, and consider finally the tracing produced by another mobile point on that straight edge.

"We can also pull two weights simultaneously from the same plane, whether their distances are fixed, or variable, during the motion in the plane. We can also imagine two planes, one in which point C is invariably linked to one plane, and the other where the engraver's point B is tracing with a very light contact (which does not interfere with the motion of B) and describes a new curve, and then suppose that this second plane has its own motion; the tangents to this new curve would be nothing else but the straight line which gives direction to the composite motion of the engraver's point in the fixed plane, and the motion of the other plane. We shall determine the properties of the tangents from this new curve so described.

"By meditating on the extreme generality of this type of motion and on the incalculable applications that it can offer, I have darkened many a paper, years ago, while meditating on their practical applications, with respect to the wonderful resources that I was noticing with the conversion of tangents, and even more within the quadratures. Having discovered a {construction} which spontaneously extends, in a totally universal manner, to all of the quadratures, I don't even know if, on this question, the latest developments in Geometry have established a more general one. I have resolved myself to publish it. Since up until now I had reserved it, as raw material, for a future work, and with the idea of developing an integral theory, my other tasks of totally different interests have taken me so much that I must, at the first opportunity, discharge myself of these old ideas for fear of loosing them; these current results, which I have waited twice as long as the time recommended by Horace, I have waited for {Lucine} long enough. (9)

"I would like to show, next, that {the general problem of quadratures is determined by constructing a curve whose inclination obeys a given law}, that is to say, against which the assignable sides of the characteristic triangle have between them a given relationship; after which I will show that {we can construct this curve by the motion that I have imagined}. In fact, for any curve C(C), I imagine {two self-similar characteristic triangles}, one assignable TBC, the other, unassignable GLC.

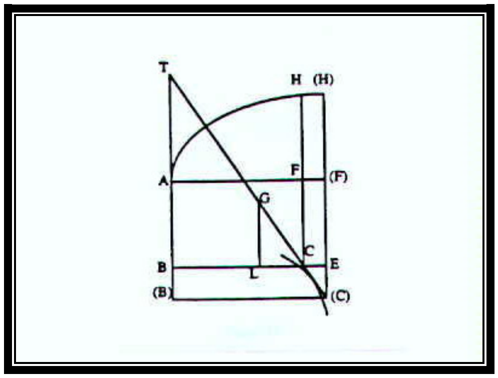

Figure. 10 "Two self-similar characteristic triangles."

"The {unassignable} triangle is delimited by the elements GL and LC of the coordinates CF and CB which form its sides, and by the elementary arc GC, which constitutes its base or its hypotenuse. But the assignable triangle TBC is delimited by the axis, the ordinate and the tangent, and expresses consequently the angle that the direction of the curve makes (that is also the direction of its tangent) with the axis or its basis, in other words, the inclination of the curve at the given point C.

"The task, here, is to square the surface F(H) located between the curve H(H), the two parallel straight lines FH and (F)(H), and the axis F(F); take on this axis a fixed point A, and take an a conjugated axis the straight line AB perpendicular to AF, and going through A, then take on each straight line HF (which can be extended at will) a point C, that is let's construct the curve C(C) in such a way that, once you have drawn between C and the conjugated axis AB (extended if need be), the conjugated ordinate CB (equal to AF), but also the tangent CT, the portion of the axis TB, that they delimit should be with BC in the proportion of HF with the constant a, that is to say, such that the product of a by BT is equal to the rectangle AFH (which circumscribes the triline AFHA).

Under such conditions, I say that the product of a by E(C) (the difference between the ordinates FC and (F)(C) of the curve) is equal to the surface F(H); from this fact, if we extend the curve H(H) all the way to A, the triline AFHA of the figure to be squared is equal to the product of the constant a by the ordinate FC of the squaring figure. My calculus shows this immediately. Suppose in fact that AF = y, FH = z, BT = t, FC = x, then you get by hypothesis t = zy/a, but also t = ydx/dy, the expression of the tangents according to the formulas of my calculus, so: adx = zdy and consequently ax = Function of zdy = AFHA. The curve C(C) is therefore a quadratrix with respect to the curve H(H), because the product of its ordinate FC by the constant a is equal to the area under the curve, that is to the summation of the ordinates H(H) applied to their respective abscissa AF. (10)

"Furthermore, since BT is to AF in the same proportion as FH is to a (by hypothesis), and since the relationship of FH to AF (relationship which determines the figure to be squared) is given, we shall also know the relationship of BT to FH, that is to say, the relationship of BT to BC, and therefore, the one of BT to TC, in other words, the relationship

between the sides of triangle TBC. That is why, the task of determining all of the quadratures and all of the measurements are realized, once you are able to establish the relationship of the sides of the assignable characteristic triangle TBC, in other words, from the law of the inclines, when you trace curve C(C), since we have shown that it is a quadratrix.

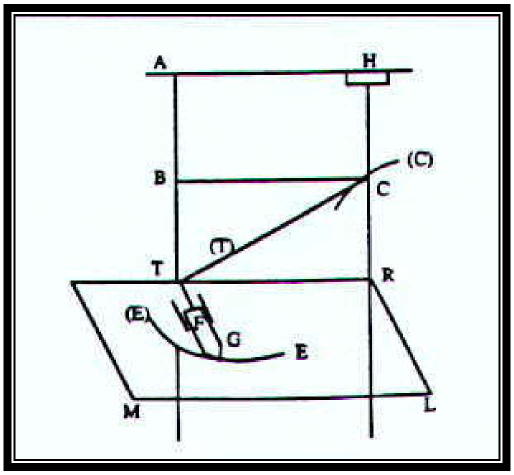

Figure 11. The general idea of an INTEGRAPH mechanical device.

"The tracing can be generated in the following manner: construct Figure. 11 by establishing TAH as a fixed right angle located in the horizontal plane; have an empty cylinder TG move along the side AT vertically under the stated horizontal plane. Inside of this empty cylinder, place a second filled cylinder FE, mobile from top to bottom, and which is connected at the top of F to a string FTC in such a manner that the part FT is located inside of the empty cylinder while the part TC is located in our horizontal plane. Furthermore, point C which describes the curve C(C), at the end of the string TC, must be bound to the plane by means of a weight which will rest on it; but the beginning of the movement will be located in the empty cylinder TG which, while coming away from A along AT will attract C. And while the point traces the curve, that is to say the engraver's point C will move ahead toward A, a ruler HR located in the horizontal plane which is perpendicular to AH (the other side of the fixed right angle TAH), this impulsion would not impair the motion of point C, which is moving solely by the traction of the string, and would therefore allow it to maintain the direction of its motion. Let us suppose, as well, the existence of a shelf RLM which is moving downward at right angle with respect to the ruler HR, at the common point R, and which is otherwise constantly pushed by the empty cylinder, in such a way that ATHR forms a rectangle. Let us suppose finally that on this shelf is traced (on a thin relief sheet) a rigid line E(E) on which the filled cylinder bites continuously through a notch which you can imagine is being engraved in the extremity E, such that as R approaches T, the cylinder FE descends. Since the length ET + TC is given (It includes the filled cylinder EF and the totality of the thread TC), and also the relationship between TC and TR or BC (because the law of the inclines is given), we will also know the relationship between ET and TR, the ordinate and the abcissas of line E(E), then we can determine, by means of ordinary geometry alone, and trace it on the shelf LRM.

"Thus, this device gives us the tracing of the curve C(C). So, by the very definition of our motion, TC is everywhere tangent to curve C(C). Here, then, is the construction of a curve C(C) whose inclinations obey a given law, that is to say where is given the relationship between the sides of the assignable characteristic triangle TRC or TBC. This curve being a quadratrix of the figure that was to be squared, as I have just shown it, we will also obtain the quadrature by the same required measure.

"We can apply similar procedures in varied ways to different problems posed by the method of inversion of tangents: for example, if point T were to be displaced on a curve T(T) (as opposed to straight line AT), the coordinate HC would have intervened in the calculation as well (that is to say the abscissas AB). In point of fact, the whole problem of inversion of tangents can be reduced to a relationship between three lengths, that is, the two coordinates CB and CH and the tangent CT, or other similar functions.

"However, we can end the discussion by a motion which is much more simple. For example, if the relationship between AT and TC had been known, (that is to say, given a family of circles, given in ordered position, and cutting a curve at right angle, find that curve), a much simplified apparatus would have sufficed. (11) Since nothing concerning H and R are to be considered, all you need to do is to trace a rigid directrix line E(E) in a fixed vertical plane passing through AT, that is to say, once the empty cylinder TG is determined, when the filled cylinder TE comes down as required by the directrix line E(E) on which it is biting, because the summation of ET + TC is constant (as before) and that the relationship between AT and TC is given, we will easily discover the adequate relationship between AT and TE, that is to say, the nature of line E(E) that permits you to find the curve C(C) which we were looking for."

NOTES:

(1) The **{AMBIGUITY OF THE MULTIPLICITY},** ***exemplified by the case of the {*****PYTHAGOREAN}** ***discovery of the functional diagonal of the square, as exemplified by the Meno dialogue of Plato, and the incommensurability singled out by*** **{NICHOLAS OF CUSA},** ***between rational and irrational numbers, and between the irrational numbers and the transcendental nature of the circle, are the most exemplary cases of this point. Kaestner, similarly, developed this idea of increasing powers between the line, the surface and the volume.***

(2) As a science of the infinite, constructable only by multiply-connected curvilinear action **({EVOLUTE/INVOLUTE OSCULATION, INVERSION OF TANGENTS, AND TRACTION}),** ***this Leibnizian*** **{GEOMETRY OF MEASUREMENT}** ***is a first attempt at defining a transfinite ordering of mathematical magnitudes between*** **{DIFFERENTIATION*****}*** ***and*** **{INTEGRATION}.**

The distinction between the **{DIFFERENTIAL CALCULUS}** ***and the*** **{INTEGRAL CALCULUS}** ***is therefore, found in the nature of the infinite itself, that is, in the change of***

*magnitudes in {*INFINITESIMALS}. *In both cases, the* {CALCULUS} *involves such* {INFINITE MAGNITUDES}. *It is a* {CALCULUS OF INFINITIES}, *either in the form of locating extremely small intervals of change within a surface, or by making summations of totalities of such infinitesimal intervals forming such surfaces. In other words, the Leibniz* {CALCULUS OF INFINITESIMALS} *is the means of solving Plato's Parmenides Ontological Paradox of the* {ONE} *and the* {MANY}. *On the one hand, the* {DIFFERENTIAL CALCULUS} *seeks to measure the* {MANY}, *in the form of infinitely small differences within intervals of change, which are unassignable to any finite number; and such intervals are called* {DIFFERENTIALS}. *On the other hand, the* {INTEGRAL CALCULUS*} is the opposite of the first; that is, it seeks to determine the* {ONE}, *in the form of a complete summation, or a totality of such non-numerical* {INFINITESIMALS}, *which are all non-linear in the small, and which form entire and complete wholes without leaving any linear residues.*

Thus, in the Leibnizian conception, neither {DIFFERENTIATION} *nor* {INTEGRATION} *need to be reduced to linearity in the small, as this will later be done by the truncations and bowdlerization of LaPlace and Cauchy. The Leibnizian* {INTEGRATION} *permits you to measure a definite area under, or above, a curve directly by means of a thread which* {OSCULATES THE INVOLUTE} *while it is tangent to its* {EVOLUTE}; *and the* {DIFFERENTIATION} *permits you to measure indefinitely small intervals of change,* {DIFFERENTIALS}, *between any two infinitely close* {OSCULATING RADII} *determining the curvature of the* {INVOLUTE}. *So, the* {QUADRATURE} *of an area does not need to be subjected to any form of Cauchy limit theorem, or some* {INDEFINITE MULTIPLICITY} *(i.e. algebraic irrational), or false integration by a* {POLYGONAL EVOLUTE}, *in the manner of LaPlace.*

For the first time, in mathematics, the discovery of the calculus can express processes, and curvilinear changes within {INFINITESIMALS} *of physical space-time, by means of non-linear, non-finite, and non-numerical magnitudes, and the Leibniz* {CALCULUS OF INFINITIES} *is the algorithm for that new mathematical physics.*

In his early manuscripts, Leibniz summarized what he called the fundamental principle of this calculus, both in terms of {DIFFERENTIATION} and {INTEGRATION}, or {SUMMATION}, *in the following way.*

"{THE FUNDAMENTAL PRINCIPLE OF THE CALCULUS.}

"Differences and sums are the inverse of one another, that is to say, the sum of the differences of a series is a term of the series, and the difference of the sums of a series is a term of the series; and I enunciate the former thus $\int dx = x$, *and the latter thus,* $d\int x = x$.

"Thus, let the differences of a series, the series itself, and the sums of the series, be, let us say,

Diffs.		1	2	3	4	5	........dx
Series	0	1	3	6	10	15	 x
Sums	0	1	4	10	20	35	..$\int x$

"Then the terms of the series are the sums of the differences, or $x = \int dx$; thus, 3 = 1+2, 6 = 1+2+3, etc.; on the other hand, the differences of the sums of the series are terms of the series, or $d\int x = x$; thus, 3 is the difference between 1 and 4, 6 between 4 and 10." **J. M. CHILD, "{THE EARLY MATHEMATICAL MANUSCRIPTS OF LEIBNIZ,}"** *The Open Court Publishing Co. Chicago, 1920.*

*Thus, Leibniz generates his calculus as a self-developing process where the sums acts as a mirror image to the differences; as in the case where a series expresses, at the same time, {***THE DIFFERENCES OF THE SUMS AND THE SUMS OF THE DIFFERENCES}.**

*(3) The construction of Roberval's cycloid will be found in {***APPENDIX I}.**

*(4) Leibniz is introducing, here, the elements of construction of a new motion, which had never been considered in Geometry before, and which he established as {***UNIVERSAL},** *as well as* **{RETARDED}** *by some physical law, and which he calls {***TRACTION}.**

*Furthermore, Leibniz develops different transfinite levels of geometry. On the one hand, the first level, which he identifies as the {***GEOMETRY OF DETERMINATION},** *is more appropriately identified as the Euclidean domain, that is the linear (i.e. straight line) algebraic domain. On the other hand, the* **{GEOMETRY OF MEASUREMENT},** *is applied to different degrees of physical retarded motions, including {***PENDULUM MOTIONS}, the {EVOLUTE-INVOLUTE MOTIONS}, and the {TRACTION MOTIONS},** *which represent the most universal forms determining the transcendental anti-Euclidean domain, that is, the domain of non-linear, non-constant curvature of physical space-time, which will later be pursued by Carnot and Monge at the Ecole Polytechnique, and subsequently by Gauss and Riemann in what will become known as hyper-geometry, or the theory of modular functions of multiply-connected manifolds.*

*(5) This is an extension of the previous methods of measurement of {***INVOLUTES}** *by means of {***EVOLUTES}** *that Leibniz had developed up until about 1693, with his calculus. So far, Leibniz has supplied two complementary methods of generating transcendental curves: one is the famous {***EVOLUTE THREAD}** *method of Huygens, the second is the method of {***OSCULATING CIRCLE BY INVERSION OF TANGENTS}.** *Now, Leibniz develops a third method which will become known as the universal motion of {***TRACTION},** *which, in a sense, is a variation on his method of {***INVERSION OF TANGENTS}.**

The general mode of generating motion by **{TRACTION}** *that Leibniz considered a new* **{UNIVERSAL METHOD}** *for the construction of quadratures is attractive because of the presence of a* **{CONSTANT EVOLUTE TANGENCY FUNCTION}***; every other parameter, such as multiple directionality of the traction, increase or diminishing length of the radius of traction, degree of weight on the stiletto, etc., are all subject to change. This will generate a universal curve which was also discovered independently by Huygens, and which he called the* **{TRACTORIA},** *that is, what will later become known as the* **{TRACTRIX}.**

Indeed, a few months earlier, in February 1693, Huygens published in the History of the Works of Scientists [Journal of Rotherdam] a construction of **{QUADRATURES BY TRACTION}** *which he described as a simple mechanism of pulling in the horizontal plane "a point, which by its weight or by other means, will offer some resistance at the end of a thread, or attached to an inflexible rod, and which will simply be pulled forward from the other end"..."If this description can be considered geometrically exact, by mechanical standards," writes Huygens, "we should consider that, what we have here, with the quadrature of the hyperbola, is the perfect construction of all of the problems that can be reduced to this type of quadrature." [Huygens*, **{OEUVRES COMPLETES}**, *Vol. X, p.407.]*

(6) This can be illustrated by comparing the construction for the **{TRACTRIX}**, *and the construction for the* **{HYPERBOLA}**. *The connection between the two is explicit in the comments made by Leibniz. Also, Leibniz implies the direct connection between the* **{TRACTRIX} and the {CATENARY CURVE}**; *that is, where the* **{TRACTRIX**} *is nothing but the* **{INVOLUTE} of a {CATENARY -EVOLUTE}.**

(7) There is definitely some confusion in the French translation of Parmentier, as well as in the Gerhardt Latin original. First Gerhardt refers to the previous Figure 8 of the **{TRACTRIX}** *instead of Figure 9, and then, relates AB and AC (and not BD and DA as Bernoulli does in the Acta of May 1693). Parmentier further confuses the issue by wrongly relating AB to AD as being the proportionals in question, which conforms neither with the Leibniz conception, nor with the Bernoulli idea. Since the text of Leibniz is rendered too erroneous because of these two translating mistakes, I propose that the text should be brought closer to the text of Bernoulli that Leibniz is relating to.*

The problem that John Bernoulli submitted to the Acta Eruditorum of May 1693 was as follows: "Find the curve ABC whose property is as follows: once you have traced a tangent from an arbitrary point to the axis AE, let the abscissa AD, and tangent BD be in a constant proportion, as in the ratio of M to N." And Bernoulli added the following comment: "Whatever the ratio of M and N, one can always describe the curve ABC by a constant motion with the same facility, independent of the fact that, with respect to the ratio of M to N, the curve can become more or less complex; in fact, in the case of equality, we can see immediately that the curve ABC is a circle."**{OPERA JOHANNIS BERNOULLI}**, *T. 1 p.65] This approach of proportionality in the construction of a curve represents the very core of the self-development quality of the Leibnizian calculus.*

In other words, Leibniz sees in the relationship between M and N, the continuity of a proportional motion, the **{REASON}** *(ratio) which underlies the whole process of generating the curve; the same* **{REASON}** *which will lead him to discover the relationship between the logarithms of the* **{CATENARY}** *curve, and of the* **{LOGARITHMIC}** *curve. (See Leibniz's Catenary and Logarithmic Curves, in* **{FIDELIO}**, *Spring 2001.)*

(8) Leibniz will go a step further, and conceive of an apparatus with which you can generate quadratures of not only an infinity of curves, but of varying degrees of infinities, **an {INTEGRAL GENERATING MECHANISM}**, *a sort of* **{INTEGRAPH}, as he calls it.** *This* **{INTEGRAPH**} *is based on a method of creating a design, in the sense of a machine-tool*

design, for the purpose of generating a new technology for the production of surfaces: an apparatus which would {COMPOSE AND MEASURE} *directly areas under curves, and under surfaces (without the use of any limitation theorem, or any type of Cauchy fraction), that is, which would determine, and measure at the same time, entire quadratures of areas with curvilinear action only, that is, without leaving any linear residues of error in the composition of their totalities.*

About this {INTEGRAPH}, *which may never have been physically constructed, Leibniz later said: "As far as my general construction of quadratures by traction is concerned, it is sufficient for it to be theoretically exact in order to be scientific, even if it were not practicable in reality. Most geometric constructions are of this nature, when they are initially elaborated.*"{C.I. GERHARDT, MATHEMATISCHE SCHRIFTEN} *II p.181] This problem which is derived directly from his investigation into the* {INVERSION OF TANGENTS} *has direct bearing on the underlying problem, which is posed by the reciprocity between* {DIFFERENTIATION} and {INTEGRATION}, *that Leibniz is approaching here already from a hyper-geometric standpoint.*

So, the concept of the {INTEGRAPH} *is not as complicated as it appears to be: it simply consists of extending the method of zero curvature traction to other non-linear motions, and to other dimensionalities; for example,* {INTEGRATING NON-LINEAR TRACTION} *into several different directionalities of action, all at once.*

Thus, the aim of Leibniz is not to construct the {INTEGRAPH} *as a machine for making quadratures, in the strict sense of a mechanical device like the construction of a calculating machine of some sort, but rather, to establish the principle of calculating transfinite quadratures, by developing a construction for adding a higher dimensionality of circular action to another: that is, to* {DETERMINE A NON-LINEAR MODULAR FUNCTION}. *Indeed, it would be silly to think that a mere mechanical device could ever replicate the complex movement of a single planet in the solar system, with or without the presence of an apparent rising Sun in the East. It was Leibniz himself who said that* {THERE IS ALWAYS MORE IN NATURE THAN CAN BE DETERMINED BY GEOMETRY}. *On the* {INTEGRAPH}, *see Gerhardt, M.S. VI p.231* {GENERAL RULE FOR THE COMPOSITION OF MOTIONS}.

(9) In his {POETIC ART}, *Horace recommended that a writer wait nine years before publishing his works. "Twice as long" means that a total of 18 years brings the original works of Leibniz on these questions back to 1674, which is a few years after he had begun to develop his calculus.*

(10) Note on the geometry of infinitesimals as indivisibles. {R.E. DE GEOMETRIA RECONDITA}. *P.141.*

(11) This construction of the tractrix shows how a family of circles cuts the curve at right angle. {APPENDIX II: A LEIBNIZ-LAROUCHE METHOD OF GENERATING THE CATENARY-TRACTRIX} *shows how to integrate the three methods of inversion of tangents, traction motion, and evolute-involute motion.*

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8

A NEW MODE OF APPLICATION OF THE DIFFERENTIAL CALCULUS TO DIFFERENT POSSIBLE CONSTRUCTIONS OF A CURVE FROM A PROPERTY OF ITS TANGENTS, by G.W. Leibniz, Acta Eruditorum, Leipzig, July 1694.

"I remember having mentioned, here, that there exist curves, which are formed by intersections of curves, exactly in the same sense that there are those which are formed by intersections of straight lines. But, I would like to explain in more details this question, which is essential for the enrichment of Geometry, because even when one deals only with straight lines, one does not grasp the full implications of its significance. I shall, therefore, show how to bring the following problem to the laws of ordinary Geometry: {given a series of tangent lines in ordered position (straight lines or curved lines), find the curve tangent to them} or, which comes back to the same, {find the curve which is tangent to an infinity of lines given in ordered position}. The applications of this problem being very extensive, I have already imagined an appropriate calculus for this question, or rather, I have applied to it my differential calculus in a very specific way, and thus provided for an appreciable gain of time.

"Just as Descartes did when he included in his calculations the locus of the Ancients, and has introduced equations which included each point of a curve, I am using here much more extensive equations which include any arbitrary point of any arbitrary curve, in the midst of a {series of successive curves}. In this way, x and y do represent the abscissas and the ordinate, that is the {coordinates} of some curve, as I have stated, but more specifically, by a sort of {expedient equivocity of the characteristic}, they apply to the curve formed by their intersections, that is to say to the curve which is tangent to them. As for the coefficients a, b, c, associated with x and y to form the equation, they represent {constant} values for a given curve, some are {inherent} (the {parameters}) while others, which are {extrinsic}, determine the position of the curve (and consequently of its summit as well as of its axis). But, when you compare the curves mutually within the series, that is to say, when you consider the passage of one curve to another, certain coefficients are {absolutely constant}, in other words {permanent} (those which remain constant not only for one curve but for all of the curves of that family), while the others are {variable}. (1)

"Obviously, in order to determine the law for {a series of curves}, there must be among the coefficients only a single variability. If, however, there appears to be several variables in the {first equation} (valid for all the curves and expressing their common character), we must further have {auxiliary equations} expressing the relationship of dependency between the variable coefficients, and thus permit the elimination of all of the variables, from the first equation, except one.

"So, when two neighboring lines concur, and their intersection determines a point on the sought for curve (we also know that this last curve is tangent to them), the concurrent

lines, also tangent to the curve of their intersections, are evidently {dual}, but the point of intersection, that is the point of concurrence, is {unique}. Consequently, the same thing applies to the ordinate which corresponds to it, while ordinarily, in the usual investigation of the tangents to a curve given from its ordinates, whether we are dealing with straight lines or curves (for example circles, parabolas, etc.), we consider that there are {two} ordinates, and that the tangents are {unique}. (2) This is why, in our present calculus where, contrary to the usual calculation, we are seeking the ordinates themselves, starting from the successive tangents, rectilinear or curvilinear, given in position, it is the ordinates x and y which remain unchanged in this passing (from one tangent to another neighboring one), and which are thus indifferentiable. As for the coefficients (which are considered as indifferentiable, in the ordinary calculations, because they are constant), they are differentiated, to the extent that they are now variable.

"We must note that {if all of the intrinsic coefficients were permanent}, and thus the concurrent successive curves were congruent two by two, nothing could stop us from considering them as the different {traces of a same mobile line}, the curve constituted by their intersections, which remains, for the entire duration of their movement, tangent to the line which is moving. This case has a certain kinship with the generation of {trochoids}; in point of fact, the basis upon which rolls the generating circle of a trochoid is also tangent to that generating circle during the movement.

"Construct the calculus in the following manner: take a fixed right angle, and consider their sides of indefinite lengths, as two axis for determining curves, that is, as an axis with its conjugate axis; the perpendiculars dropped from some points of a curve onto these axis will be ordinate x, and the conjugated ordinate y, the abscissas, in other words {the coordinates}. By looking to discover the relationship among those coordinates, we will get {equation} (1) that I will call first equation, because it is common to all of the points of all of the ordinate curves.

"If equation (1) has several variable coefficients, b, c, their relationship of dependency will be given by one or several {second} (2) equations, such that by eliminating from equation (1) all of the variable coefficients except one, b, we will get equation (3). The differentiation of this last one will lead to equation (4); but since from this last one, there remains only the differential of b, the differentiability vanishes (3), and equation (4), which we got, becomes an ordinary equation. This equation allows us to eliminate the last variable b from equation (3), which will give equation (5), in which, except for x and y, there remains only invariable coefficients (as a), such will be the equation for the sought for curve, which is formed by the intersections of a series of lines, and therefore the equation for {the series of tangent lines which are common to it}.

"But we can also construct this calculus in a different way, if it turns out to be easier, by not eliminating all of the variables right away, but by keeping them. From the first equation (1), and from the second equations (2) (we must keep a sufficient number of them in order to make explicit the links between the variable coefficients), lets differentiate equation (1), which will give equation (3), and lets differentiate equation (2), which will give equations (4) (depending on whether (2) included one or several equations). We acquire in this way,

several differentials, but with an equally sufficient number of equations to eliminate them, and naturally, as soon as we are able to eliminate all of the differentials, except one, this last one will disappear by itself and will give us the equation (5), which will be an ordinary equation, that is to say, where no differential is involved; by composing this last one with the equations (1) and (2), we should be able to eliminate all of the variables, and get the equation (6) expressing the nature of the sought for curve, that is formed by the concurrence of the curves. This equation will be identical to the equation (5) of the preceding calculus.

"This method permits us to resolve numerous problems of superior Geometry which were escaping us, up until now, and which deal with the inversion of tangents. The following are a few general examples. For instance: given a relationship between AT and A-π, the portions of each of the two axis delimited by the tangent CT to a curve, find the curve CC.

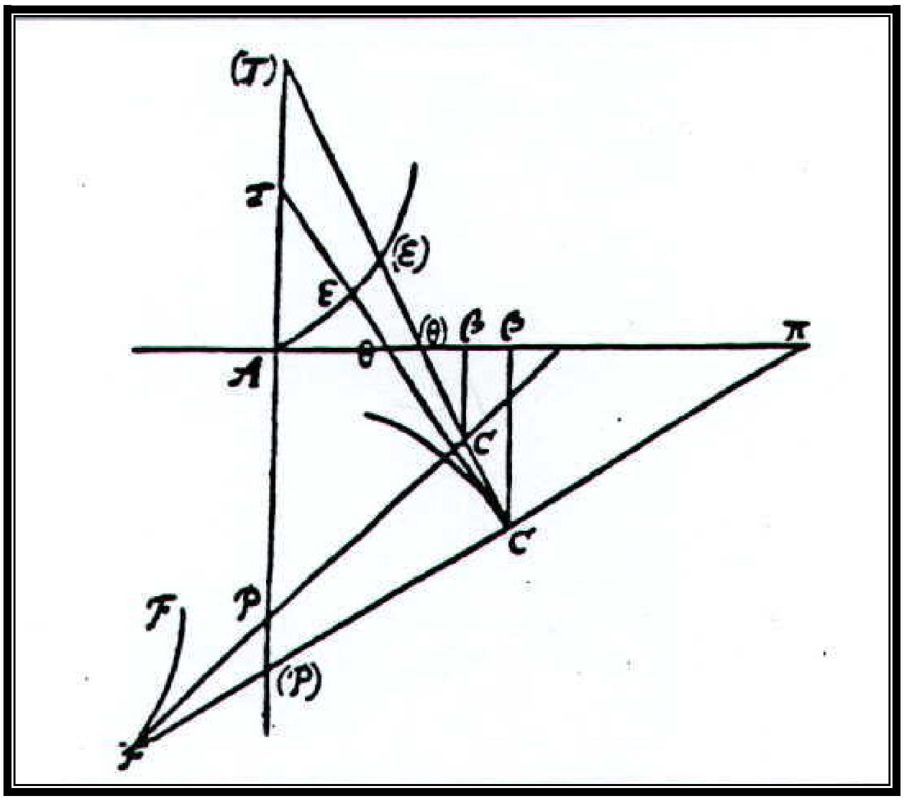

Figure. 12 Curve CC found by inversion of its tangents.

"Given that tangents in ordered position can be generated from a known curve, the same can be said of the curve formed precisely by their concurrence. In other words, given a point T on the axis, and point E on a curve E(E), such that the segment TE, which can be extended, is also tangent to the sought for curve CC, it is clear that, following what was just stated, the indicated method can lead you to obtain curve CC.

“Similarly, given the relationship between the two portions of the axis AP and A-pi delimited by the perpendicular PC to the curve, we can find curve CC: the succession of straight lines P-pi given in position are known, and so is given curve FF which is formed by their concurrence, and by developing this last one, we describe curve CC that we were looking for. In this last case, we can even get an infinity of such curves which satisfy the conditions of the problem, all ({involutes}) curves parallel to each other, and {co-described} by the same development curve ({evolute}); and furthermore, not only can we find the sought for curve from the relationship between AP and A-pi, but, we can find another curve which {passes through a given point}. In such a case, however, CC is not always an ordinary curve, because as the curve formed by the concurrence of straight lines given in position, it is not itself given, but only produced by its generating curve. Of course, with this curve which is formed by the concurrence, we get a determined curve, and it is not possible to locate a point through which the curve goes through, a distinction which has its usefulness in this present theory. (3)

“I will give and example of this calculus, in a problem just as general, but which is applied to a particular curve: given the relationship between the normal PC and the segment AP which this perpendicular cuts on the axis, find the curve CC.

“Evidently, if we consider circles centered at P, and whose radii PC, are of known lengths (we know their relationship to AP), we get a {family of circles tangents to curve CC} (the Tractrix curve) from which we can also deduce the curve formed by their concurrence; I have already discussed this matter in the past at the end of my article published in the Acta of June 1686, p.300. (4)

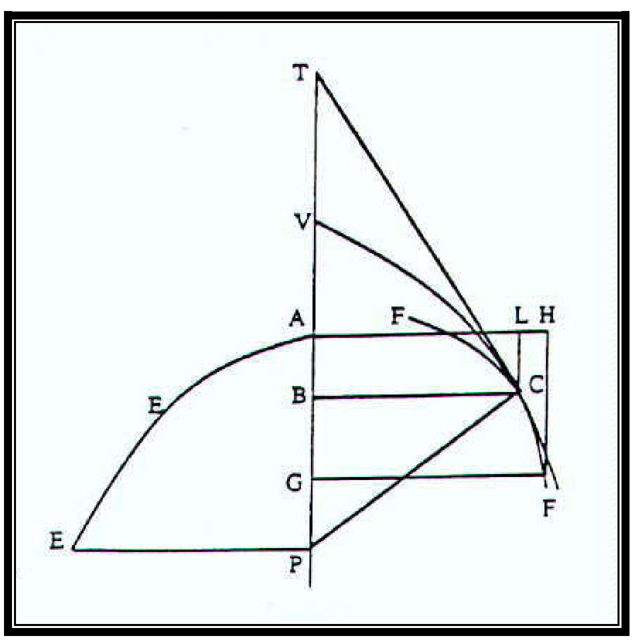

Figure. 13 "Circle CF of center P."

"Let us trace circle CF of center P and of radius PC, whose length is known. Now, in order to apply the method that I have just proposed, draw from an arbitrary point F of the circle perpendicular lines to the right angle sides of PAH, that is to say, the coordinates FG = y and FH = x (which at the intersection point of two circles end up in CB and CL), let us pose AP = b, and PC. We shall have the definition of the circle: xx + yy + bb = 2bx + cc, a first equation (1), common to all of our circles and to all of their points. But, since we also have at our disposition the relationship between AP and PC, we get curve EE, whose ordinate PE is equal to PC. Suppose (for example) that you are dealing with a parabola of parameter a, and that we have ab = cc, this second equation (2) makes the relationship explicit, that is the link of dependency, between c and b. By eliminating c from equation (1), we will have xx + yy + bb = 2bc + ab; obviously equation (1) contains, on top of the coordinates x and y, the coefficients c,b,a; and among them c and b are constant only for one given circle, c being internal to the circle, and represents the radius, b is external, because it indicates the position of its center; both of them are variables when the circles change, while a is absolutely constant, or permanent, because it remains the same for all of the points of a same circle, as well as for all circles.

"When equation (3) is brought to the level of a single variable coefficient b, there remains only to differentiate it according to b (the only differentiable quantity that it has), which gives us 2bdb = 2xdb + adb, that is (db vanishing): b = x + a/2 (in which case when only one differentiable magnitude exists, the calculus is reduced to the old method of maxima and minima that {Fermat} had introduced, and that {Hudde} had promoted, and which is but a corollary of my method). Now then, by eliminating from equation (3) the only remaining variable coefficient, we get equation (4), where ax + aa/4 = yy which is the equation of the sought for curve CC. This shows that you are dealing with a parabola, which is congruent with the first parabola AE, but only slightly displaced, since its summit will intersect the AP axis, above the summit A of parabola AE, in such a way that the distance AV between the two summits corresponds to the quarter of their common lateral side. If you prefer the other way of calculating, bringing in many differentials, repeat equations (1) and (2) and differentiate them, (1) will be bdb = + db + cdc, but (2) will be adb = 2cdc, while in equations (3) and (4) dc will vanish together from the last equations, as well as db at the same time, and we will get b = x + a/2 as we did before. So then, by eliminating the variable coefficients c and b from (1), (2), (5), we obtain, as in above, ax + aa/4 = yy for the equation of the sought for curve.

"I have then shown how to determine a curve CC from the relationship between the perpendicular PC and the segment AP which corresponds to it on the axis, because in this case, we have at our disposal a family of circles given in order and tangent to the curve. But, finding the curve CC from the relationship between the tangent TC and the segment of axis AT that the curve CC meets (that is from the perpendicular circles given in order), another method is required: we can get such a curve by means of a {tractrix construction} which I have explained, last year, in the Acta of September. (Leibniz formulated that problem in the following way: "Find the curve which is cut at right angle by a family of circles.")

"Our current method is also very useful in a number of other problems of higher Geometry, or even in the domain of application of geometry to mechanics, or to physics.

When the problem concerns the formation of a figure whose tracing must satisfy a certain condition in all of its points, we arrive at such a desired curve by forming it with concurrent lines, each satisfying the required condition at every point, that is precisely at the point of their intersection. (Leibniz developed the require condition for such a curve in his writings on {Osculation} and {ON A CURVE GENERATED FROM AN INFINITY OF LINES...})

"It is with this process that, in my article on Optical Curves, I have found a long time ago, the construction of curves which are capable of taking rays of light which are given in ordered position, project them in a mirror of a certain shape, and render them convergent, divergent or parallel. If you want to have the rays converge, then such a curve must be formed by the concurrence of ellipses; the same method applies if you want to render the rays parallel or divergent. (5)

"P.S. The solution that marquis the L'Hospital proposed to the Bernoulli problem of the month of May of last year, was inserted in the Acta, with the objection of an anonymous author, M. le marquis has well defended his own position and has demonstrated, unless I misunderstood him, that if the anonymous author had pursued his own calculus to the end, he would have found the same solution by himself. Furthermore, this anonymous has not come up with any other solution, and will not be able to do it either if he is satisfied with ordinary analysis. As for my new method, which is also the method of marquis de l'Hospital and of the Bernoulli brothers, it has not only triumphed over the problem, but also over numerous other similar problems, as it was announced in July of last year, either in an absolute and direct way, either by way of quadratures. The general problem can be formulated in the following manner: {given the ratio of two functions, find the curve.}

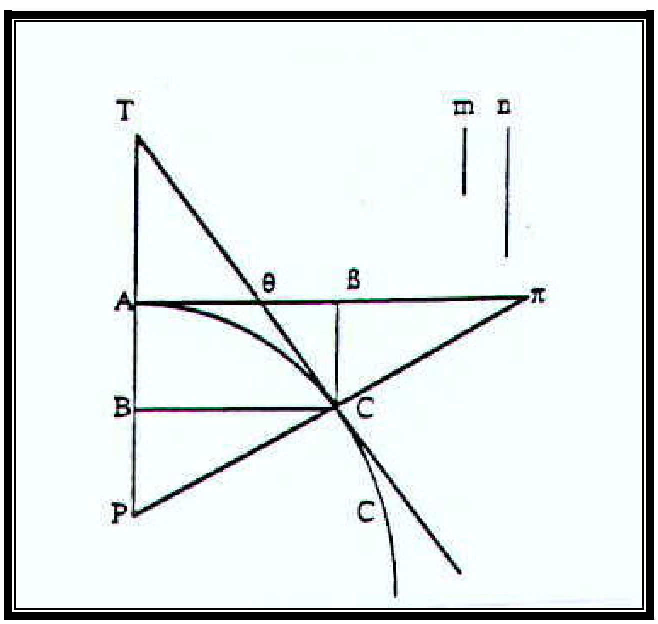

Figure. 14 "Given the ratio of two functions, find the curve."

“Let the ratio of two magnitudes be given, such as m and n. I call {function} any segment of a straight line which may be drawn from indefinite lines with respect to a fixed point, and to points of curvature on the curve. (6)

“ Such are the abscissas AB or Ab, the ordinate BC or bC, the tangent CT or C-theta, the perpendicular CP or C-pi, the segment cut by the tangent AT or A-theta, the segment cut by the perpendicular AP or A-pi, the segment delimited by PT or pi-theta, the radius of osculation, that is the radius of curvature, CP, and a lot of other things.”

NOTES:

(1) Leibniz is actually describing the non-constant curvature of the function by showing how to introduce the non-constant values into the coefficients. At a higher level, he will introduce this idea not only for finite equations, but also for infinite ones, as well as to the variation itself, which he will express as differences of differences, that is, higher degree differenciations. Integrations will correspond to theses differenciations, and so will be their reciprocals, as he says, just as in the cases where powers are reciprocal to roots. This is a lesson that Gauss assimilated and pursued in his own work on astronomy. [See Jonathan Tennenbaum, **{HOW GAUSS DETERMINES THE ORBIT OF CERES},** ***in The New Federalist, April 27, 1998.]***

(2) This emphasizes, again, the idea that the non-linear function of **{OSCULATION}** ***implies a*** **{DOUBLE}** ***contact, and consequently, a*** **{QUADRUPLE}** ***intersection.***

(3) The text of Leibniz is very difficult here because he is addressing the problem of the ambiguity of the constructive relationship between the **{EVOLUTE}** ***and the*** **{INVOLUTE}*****, between the One and the Many; that is, he forces the reader to discover the paradox of knowing the end of the process, even before he start. Indeed, how can one generate an*** **{EVOLUTE}** ***from an*** **{INVOLUTE},** ***when the*** **{INVOLUTE}** ***is the result of the generative process of the*** **{EVOLUTE}*****? Jonathan Tennenbaum raised a similar paradox with the hyper-geometric method of Karl Gauss in his determination of the orbit of Ceres. As Jonathan put it: “These functions cannot be constructed “from the bottom up,” but have to be handled “from the top down,” in terms of the characteristic singularities of a self-reflexive, self-elaborating complex domain.”[See Jonathan Tennenbaum,*** **{HOW GAUSS DETERMINED THE ORBIT OF CERES},** ***The New Federalist, May 4, 1998.]***

The ambiguity is given as a point of method here, where what is being generated also implies the principle of generating it. For example, while applying the tangents TC to determine the existence of an **{EVOLUTE CC}*****, that very*** **{EVOLUTE},** ***is nothing else but the*** **{INVOLUTE}** ***that is being generated by*** **{EVOLUTE FF}.** ***So, as an*** **{INVOLUTE},** ***it is being determined by*** **{EVOLUTE}** ***FF, and as an*** **{EVOLUTE}** ***it is being generated by the inversion of tangents TC. Here, as well as in many other locations in the Acta, Leibniz is forcing the mind of the reader to make the discovery of his method, from both ends of the process, so to speak, of its construction; that is, by discovering the principle of the constructability of his calculus, as a general theory of non-linear functions; and, at the same time by discovering its laws of change by walking through it yourself.***

The idea of discovering problems from something that has to be accomplished inversely is not new for Leibniz; and his method of application of **{INVERSION IN THE INFINITELY SMALL}** *may shed some light on the problem of* **{INVERSION}** *of weak and strong forces which comes up in the question of nuclear fusion processes.(See Bostick and Wells on the geometry of a force-free plasma) The early manuscripts of Leibniz show that the* **{METHOD OF INVERSION OF TANGENTS}** *is already a major preoccupation in his mind, as of the period of 1673-1675. He writes: "Hence, a way of describing that other curve* **({EVOLUTE})** *that touches the given curve* **({INVOLUTE})** *is generated: Now, when this is described, let the tangent be drawn at the point which is common to it and to the proposed curve, which tangents we have supposed to be already known; then this tangent will touch the given curve"*

"I think that, in general, the calculation will be possible by this method of assuming a second curve, as we have done in this case, which evidently works out one of the unknowns. Hence, I fully believe that we shall derive an elegant calculus for a new rule of tangents, which in addition may be better than that of Sluse, in that it evidently works out immediately one of the two unknowns, a thing that the method of Sluse did not do. Now, this very general and extensive power of assuming any curve at will makes it possible, I am almost sure, to reduce any problem to the inverse method of tangents or to quadratures." **{THE EARLY MANUSCRIPTS OF LEIBNIZ}**, *by J.M. Child, The Open Court Publishing Company, Chicago, 1920, P. 113.*

*An afterthought of the whole process calls for having, as a single constructive forethought: the triple method of generating the sought for curve CC; 1) by the {***INVERSION OF TANGENTS}**, *2) by an* **{EVOLUTE}**, *and 3) by means of* **{OSCULATING CIRCLES}**. *It is likely that, by internalizing this triple Leibnizian design for generating curves, in the infinitely small, or in the infinitely large, a discovery can be made which will solve the nuclear fusion problem of weak and strong forces. I don't really know how to formulate this, at this point in time, but it might be something like*:**{FIND THE FUNCTION OF DOUBLE CURVATURE, OF POSITIVE AND NEGATIVE CURVATURES, WHICH ACCOUNTS FOR THE NON-ENTROPIC ANOMALY OF WEAK AND STRONG FORCES EXHIBITED BY FUSION PROCESSES}.**

(4) Both of his articles on the **{NEW APPLICATION OF THE DIFFERENTIAL CALCULUS...}** *dated July 1694, and {***THE CURVE GENERATED FROM LINES...***} from April 1692, form the basis of Leibniz's* **{DIFFERENTIAL CALCULUS}** *with respect to evolutes and involutes; they are of particular importance for understanding the notion of* **{ENVELOPE}** *as a transcendental means of resolving the paradox of the One and the Many. These new discoveries will be at the center of the work that Gaspard Monge will further develop on the theory of* **{ENVELOPES}** *at the Ecole Polytechnique.*

(5) Leibniz develops this idea in both **{OPTICAL CURVES AND OTHER QUESTIONS}, and {THE CURVE DERIVED FROM AN INFINITY OF LINES}.**

(6) Leibniz will find that the non-linearity of a curvature is constructable whenever the reason between two functions can be found to reflect some constant property of the curve. Even though Leibniz uses straight-line segments to express functions, this does not mean that his functions

express linearity, or that all possible functions are necessarily straight lines. On the contrary, such straight lines are the reflections, the "traces", of the non-linearity of multiply connected circular action, and the principle of reason in the constructability of a curve. By virtue of his principle of continuity, whereby all lines have curvature, these straight-line segments simply express zero curvature in that situation. He wrote to Huygens, on June 29, 1694: "For my part, I find that I can always give the solution whenever a reason (ratio) is given between two arbitrary functions... Even if there was an equation in which would enter no other straight lines but these functions, no matter how many functions could be included at the same time, the curve would always be constructable." This is also how Leibniz applied the principle of proportionality for the discovery of the **{CATENARY CURVE},** ***as a logarithmic curve.***

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PART IV

APPENDIX I

THE ROBERVAL DISCOVERY OF THE CYCLOID
By Pierre Beaudry

THE PARADOX THAT BAFFLED ARISTOTLE, GALILEO, AND DESCARTES.

The history of the discovery of the **{CYCLOID},** ***which was written by Blaise Pascal, actually begins with a paradox. For centuries,*** **{SCIENTISTS}** ***have been baffled by a paradox presented by the simple rotation of the wheel of a cart. It was Aristotle who became the first known victim of this paradox, when he reported about it in his*** **{MECHANICS},** ***questions 24, and 25. The problem that he could not resolve was as follows: Why is it that, when two circles of different diameters are united together around the same center, they both travel the same distance; and when they are separated, they will be traveling different distances? Indeed, if a cartwheel makes one revolution, the hub of the cartwheel will also make one revolution: thus, the distance that the hub traveled is the same as the distance traveled by the cartwheel.***

This paradox did not escape the scrutiny of a number of astute observers, but Aristotle was the first in recorded history to fail to solve it, and so did quite a few after him, such as: Charles Bovelles (1501) A. Piccolomini, (1547), Cardan (1570), Benedetti (1585), Monantheuil (1599), Galileo (1599), Mersenne (1615), Descartes (1638), and Cavalieri (1639).

HOW ROBERVAL SOLVED THE PARADOX.

In 1634, Gilles Personne de Roberval solved the paradox of the cycloid, and discovered a method of construction which allowed him to determine 1) the true shape of the curve, and of its companion curve, the sine-curve, 2) the length of the curve, and it tangent, and 3) the area under the curve (the quadrature). This new method of construction of curves became one of the founding principles that Gottfried Leibniz established as the basis of his calculus. Roberval tackled the paradox with a simple method of **{COMPOSITION OF TWO MOTIONS}.**

Draw two concentric circles whose radii are respectively AB, and AC, in a ratio of 2/1. Rotate the circle AC, one full rotation on a straight line, from C to F. Both circles will be submitted to two different motions: one is the straight line **{TRACTION MOTION}** ***of point A along line AD, and the other is*** **{CIRCULAR MOTION}** ***of point C traveling in a curved motion from C to F. Roberval solved the paradox by simply solving the discrepancy between the two motions of the different points A, B, and C which are traveling differently, at the same time.***

Roberval is obviously forced to conclude that while C moves from C to F, in a single revolution, Point A travels across AD in the same amount of time, such that the time of CF and the time of AD are the same.

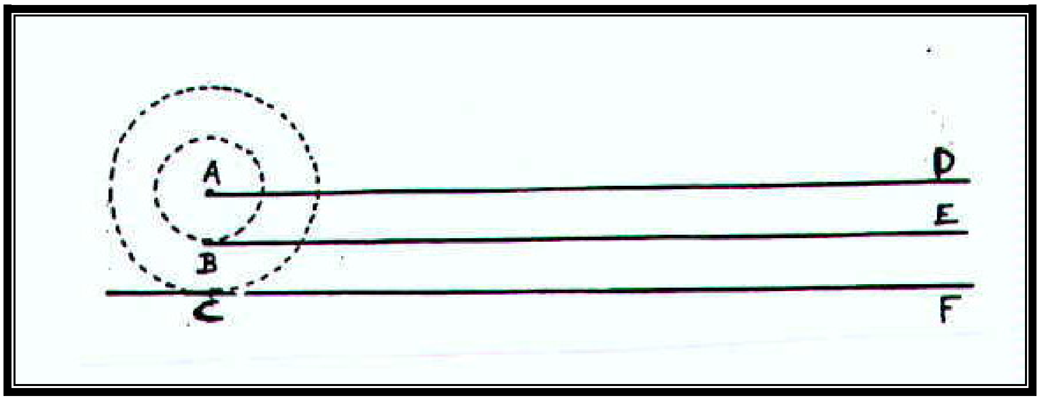

Figure 15. Two concentric circles rolling together.

However, observe the smaller circle of radius AB, which Roberval chose to be half of circle AC. Being that the time of AD is equal the time of CF, because the line AD is equal to the length of the circumference of the larger circle, the speed of A moving on AD will be double the speed of B moving in a circular motion, because the distance AD is double of the circumference of circle AB. The time of travel of point B will therefore be the same as the time of travel of point C but, because of their difference in circumference, their difference in speed will be in the ratio of 2/1: the proportionality of the circles being double. Thus, the paradox emerges when point A reaches point D, at the same time that point B arrives at point E, both after a single revolution. Thus, the time of travel of points ABC is identical in reaching the respective positions of DEF, but their speed will differ, and the difference in the speed of point B is compensated by the addition of a second kind of motion.

The paradox is solved by means of the* {COMPOSITION OF THE TWO MOTIONS}. *Because the proportionality of the spaces between the two circles is double, the same proportionality must also be accounted for between the two motions. On the one hand, in the case of circle AB, the* {TRACTION MOTION} *is acting on point B, by pulling it horizontally, in a proportion which is half the action of the* {ROTATION MOTION} *of C, that is, in proportion of 2/1, in the same amount of time. On the other hand, in the case of circle AC,* the {TRACTION MOTION*} is acting on point C, in equal proportion with the {*ROTATION MOTION}, *in the same amount of time. The reader should also note that point C will move at different speeds, while moving along the cycloid curve, while point A remains at a constant speed. This interesting new anomaly shall be explained later with Huygen's construction.

The following description comes from the original drawings of Roberval, and shows the step-by-step division of the two motions of the radius describing the different positions of the cycloid curve.

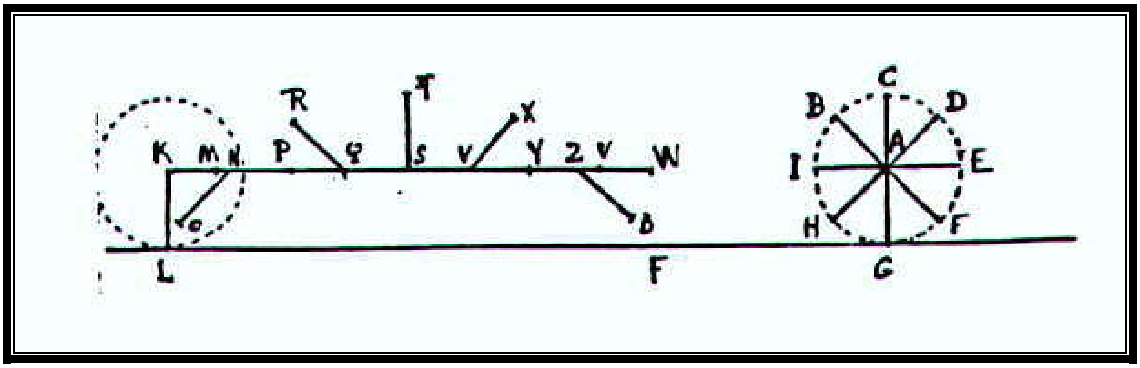

Figure 16. Parallel distribution of the radii of circle.

Roberval writes:

"In the following figure, take a circle with center K and line LF, along which it rolls and ends its conversion. Next, suppose that the straight motion as well as the circular motion are uniform. Then, describe a circle with center A equal to circle K, in a way that line LF touches circle A at point G. Then, draw two diameters GC, IE at right angles, and per continuum, by apportioning the circumference by half in a continuous proportional manner, divide that circumference in as many number of equal parts that you wish, [the line will be defined more precisely the more you have parts], let us say eight, for example, at points G, H, I, B, C, D, E, F, equal and parallel to LF, and divide KW in as many equal parts as in circle A by the points K, N, P, Q, S, V, Y, Z, W. From point K draw KL equal and parallel to AG, from point N, draw NO equal and parallel to AH, from point P, draw PM in the same direction as AI, and equal to it, then, draw QR equal and parallel to AB, from point S, draw ST equal and parallel to AC, from point V, draw VX equal and parallel to AD, from point Y, draw YV equal and in the same direction as AE, from point Z, draw ZB equal and parallel to AF, and finally, from W, draw WF equal and parallel to AG.

"I say that the line described by point L will pass through points O, M, R, T, X, V, B, and will end in F. This is the case whether the line is equal or not to the circumference of the circle, such that, if you divide circle A in as many parts as you wish, you shall find this precise description." (1)

The reader must not fail to observe here, with his mind's eye of course, something that did not escape Leibniz; and that is, when you rotate the radius KL in the described fashion, you are also transporting it, or pulling it by traction along line KW. This dual motion of {*ROTATION*} and {*TRACTION*}, is the crucial multiply connected circular principle that Leibniz will recall, over 50 years later, in his construction of the tractrix curve. So, let us just plant this little seed in the reader's mind, for the time being, and, we shall have the opportunity to see how it will grow into a fully bloomed flower, later on. In any event, this Roberval construction was not made public until after the published results of the European-wide challenge had been made public by his student Pascal, in 1658.

{THE METHOD OF PARALLEL TRACTION FROM A HEREDITARY CIRCLE}.

What is most exciting about the Roberval discovery, is its simplicity, and the profound implications that it embodies for other transcendental curves. We will show, here, how the greatest discoveries are always the simplest ones. In his {**TREATISE ON THE CYCLOID}*****, (2) Roberval showed all of the geometrical properties of the cycloid, indicating that they can be of different forms, some short, some longer, but that they are never "made up of straight lines," that is they are all {*****NON-LINEAR CURVES}*****. He discovered also {*****COMPANION}** ***of the {*****CYCLOID CURVE}, the {SINE CURVE},** ***whose base, summit, axis, and center, are all identical with those of the cycloid. He also emphasizes that straight lines cannot construct this*** **{SINE CURVE}** ***either.***

For our purpose here, we shall examine the construction of the cycloid and the sine curve more closely. Establish first that the line AC is equal to the semi-circumference AGB. If you divide this half circle into an infinity of equal parts such as AE, EF, FG, GH, etc., Next, rotate the circle, from left to right, from point A to point C. Note that, as the circle starts rolling, point A will rise and will cross each level E,F,G,H, etc., in a continuous motion, and such points will locate the sinus lines E-E1, F-F1, G-G1, H-H1, etc., as well as the sine lines M2-M1, N2-N1, O2-O1, P2-P1, etc.

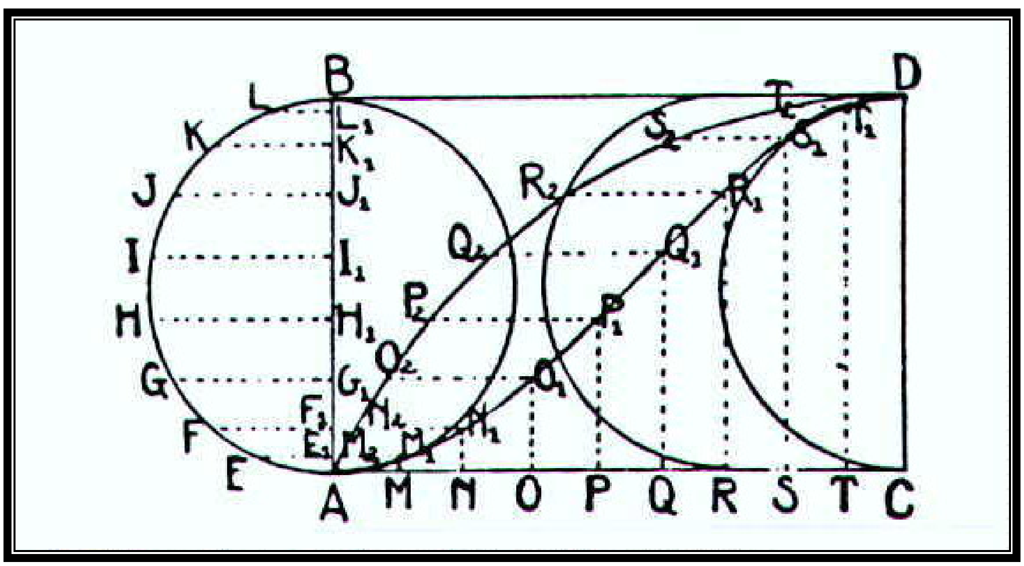

Figure 17. The are between the cycloid and the sine curve.

Divide the line AC into as many parts as you will have chosen to make on the half circumference of the circle. Starting at A, extend the different lengths AM, MN, NO, OP, etc., which will all be equal to their corresponding lengths AE, EF, FG, GH, etc., on the

circumference of the circle. You can now translate all of the sines of the half circle into a corresponding parallel position of sines between the cycloid and the sine curve.

*The discovery of the area under the {*ORDINARY CYCLOID*} by Roberval is an exquisite gem whose elegance and simplicity reside in the fact that its construction satisfies entirely the complete infinitesimal determination of the space under the curve, that is, it's complete {*QUADRATURE*}, and this, simply by means of the hereditary property of its generating circle. This is the first method of integration with the use of infinitesimals. Roberval initially identified his method, the "method of infinities," but he later came to accept Cavalieri's identification of "indivisibles." Leibniz also referred to Cavalieri's "indivisibles' as infinitesimals. At any rate, we shall demonstrate that the {*TRAITE DES INDIVISIBLES*} of Roberval is actually a {*TREATISE ON INFINITESIMALS*}, in the sense of Leibniz.*

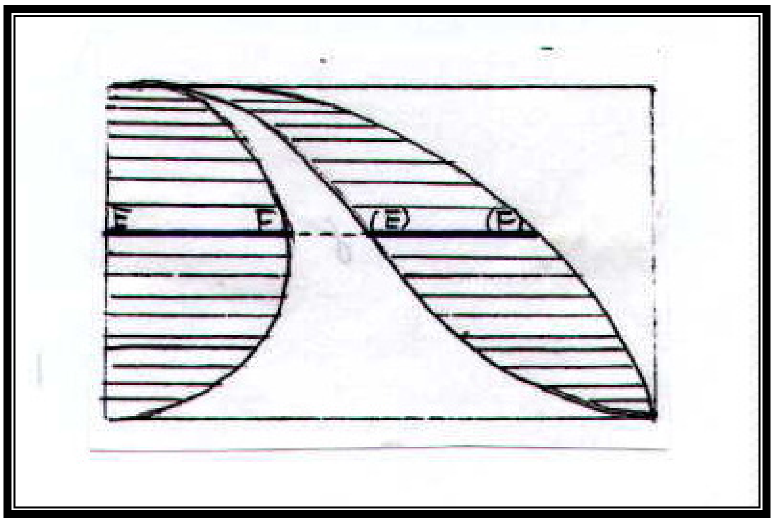

Figure 18. Simplified construction of Figure 17.

*To obtain the {*QUADRATURE*} of the cycloid, simply translate the infinity of sines EF horizontally from the half-circle to the area immediately corresponding to (E)(F) under the curve of the cycloid. Using a simple horizontal and parallel translating motion, the totality, that is, the infinity of the sines of the half circle will become the sines which fill the entirety of the space between the {*CYCLOID CURVE B(F)D*} and its companion curve,* **the {SINE CURVE B(E)D}.**

*Similarly, by translating an infinity of chords EF from the half circle AEBA into as many tangents (E)(B) of the {*CYCLOID*}, you can easily see how you can fill the entire area above the {*CYCLOID B(E)D*} with the total area of that second half circle AEBA. The reader should note that, in his Pendulum Clockwork, Huygens established that all of the chords BE, as well as all of the tangents (B)(E), are tautochronic. This means that, if a series of balls were to roll from point B, or (B), along all of those lines, they would arrive at the other end of each line, E, or (E), at the same time. This simple determination of a tangency function demonstrates that the {*CYCLOID CURVE*}, to which these* *lines are tangents, is itself a tautochronic curve. Jean Bernoulli will later show that the same ordinary cycloid curve is also the curve of most rapid descent, that is, the physical characteristic of the* **{BRACHISTOCHRONE CURVE}.**

*The reader should note at this point that the transcendental nature of the cycloid curve precluded any application of algebraic rules, such as Descartes would indulge in, for the determination of the curve. It is for that very reason, that Descartes excluded transcendental curves from the domain of Geometry altogether. It is Leibniz, however who will establish the transcendental nature of the curve in the Acta Eruditorum of June 1686, entitled {***DE GEOMETRIA RECONDITA ET ANALYSI INDIVISIBILIUM ATQUE INFINITORUM}.** *In this piece, Leibniz showed that from the standpoint of* **"{INDIVISIBLES}"** *and* **"{INFINITIES}"**, *it is impossible to express algebraically the quadrature of any transcendental curves.*

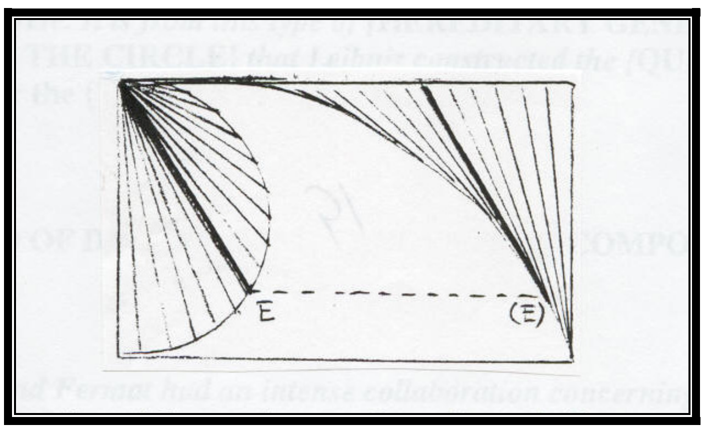

Figure 19. Construction of the quadrature above the cycloid curve.

*By having filled these two specific areas, under and above the {***ORDINARY CYCLOID},** *with the equivalent of two half-circles, it becomes a total child's play to discover that the {***QUADRATURE}** *of the area under the* **entire {CYCLOID CURVE}** *corresponds precisely to three times the area of the generating circle. Indeed, since the sine-curve divides the rectangle into two equal parts, both of which corresponds to four times the area of the generating circle, it becomes clear that the area above the entire curve is equal to one full circle, while the whole the area under the entire curve is equal to three times the area of the generating circle.*

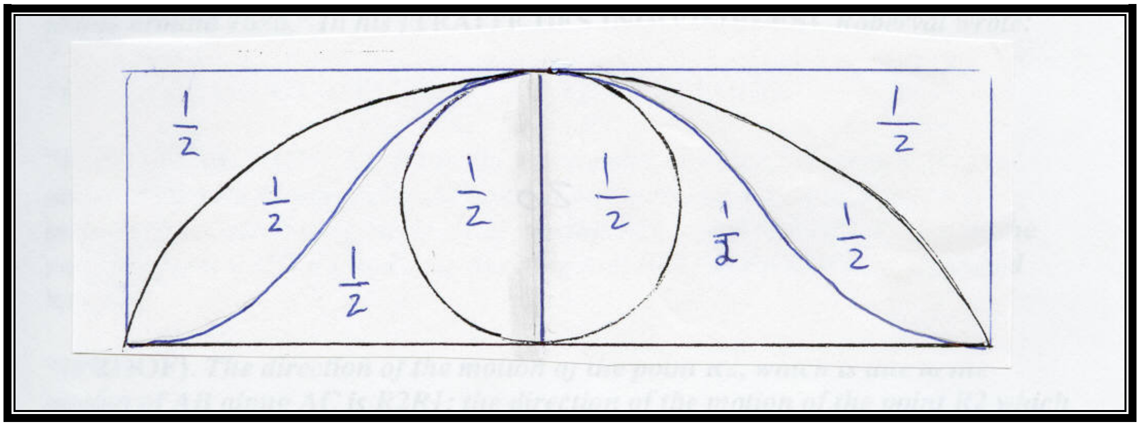

Figure 20. The area under the cycloid curve corresponds to three times the area of the generating circle.

*Note that the specific character of this method involved the hereditary function of the circle. It is from this type of {***HEREDITARY GENERATIVE PRINCIPLE OF THE CIRCLE}** *that Leibniz constructed the {***QUADRATURE} of the area under the {TRACTRIX CURVE}.**

{THE METHOD OF DETERMINING TANGENTS BY COMPOSITION OF MOTIONS}.

Roberval and Fermat had an intense collaboration concerning methods of establishing tangents to curves. The groundbreaking work of Leonardo da Vinci, and of Kepler was making the question of tangents an urgent question of physics, at that time, because the issues of gravitation, and of motion demanded a natural and efficient way of determining direction, especially the non-linear direction of planetary orbits, and of light propagation. Starting in 1635, Roberval and Fermat corresponded regularly and discussed their respective methods of generating tangents. Roberval developed a beautiful synthetic-geometric method, as oppose to an analytical one. It is consistent with the dual nature of the **{TRACTION MOTION}** *and the {***ROTATION MOTION}***, that is the dual non-linearity of the cycloid. Roberval wrote to Torricelli: "But our trochoid (cycloid) gave me an opportunity to examine the composition of motions. That was a sufficient opportunity, and, having produced a universal proposition for tangents, we made it public around 1636." In his {***TRAITE DES INDIVISIBLES}***, Roberval wrote:*

*"{***TO CONSTRUCT A TANGENT TO THE CYCLOID}.**

*"{***CONSTRUCTION}.** *Let R2 be the given point at which the tangent is to be drawn. Draw R2R1 parallel to AC, and of any convenient length. Draw R2U tangent to the generating circle RR2, making R2U equal to R2R1. Complete the parallelogram R2UVR1 and draw the diagonal R2V. Then R2V is the required tangent.*

"{PROOF}. *The direction of the motion of the point R2, which is due to the motion of AB along AC is R2R1; the direction of the motion of the point R2 which is due to the motion of point A on* *the circumference is R2U; and since these two motions are always equal, it follows that R2R1 must equal R2U. Therefore R2V is the tangent of the cycloid at R2, since it is the resultant of the two motions."*

And Roberval adds what we could call the characteristic of his method of curvature; that is, as Leibniz developed later, an infinitesimal portion of the curve representing, in the small, how the curve develops as a whole:

"{ADDENDUM}. ***If, instead of being equal, the magnitudes of the two motions have been in some other ratio, the parallelogram would necessarily have been constructed with its sides in that ratio." (3)***

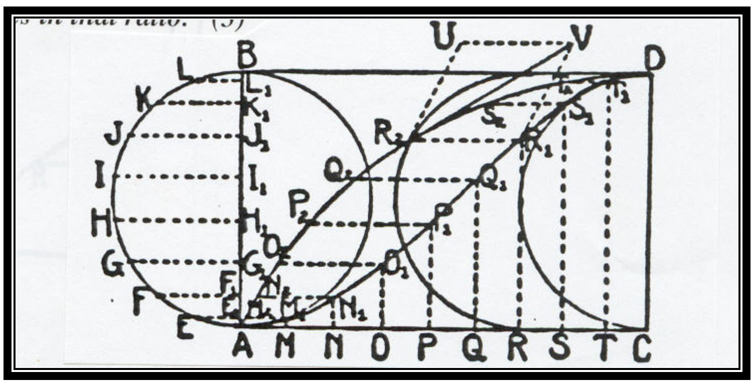

Figure 21. Tangent construction by the {COMPOSITION OF TWO MOTIONS}.

Again, this beautiful and natural synthetic construction of the tangent is derived from the same idea of the composition of the two motions. Since the cycloid is described by a point that moves both by {*TRACTION} and by {ROTATION}*, at the same time, it follows that the moving point lies on two tangents, because it lies on two different curves simultaneously.

Fermat used a similar construction. However, he emphasized the parallelism between the tangents of the cycloid with the chord of the circle, as we will do ourselves, later, as a general method of {*PARALLEL TRANSLATION} *back to the {*HEREDITARY CIRCLE}. *Fermat writes:*

"{DRAW A TANGENT TO THE CYCLOID}.

"Let HRCG be a cycloid, having its vertex at C, and the axis FC; and let COMF be the generating circle. You are required to draw the tangent to the cycloid HRCG at the point R.

"{CONSTRUCTION}. ***Draw RMD perpendicular to CF; the chord MC, and RB parallel to MC. Then RB is the required tangent." (4)***

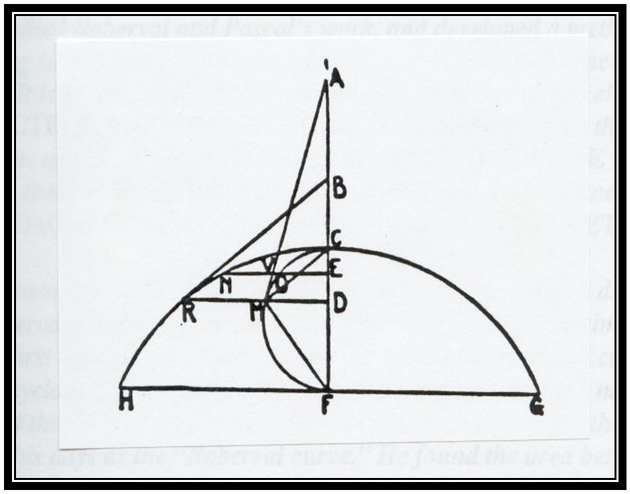

Figure 22. Fermat's construction of a tangent to the cycloid.

The method of Fermat is both synthetic and analytical. We only show, here, the synthetic construction.

It is clear that Roberval had been a student of Archimedes who defined the motion of a spiral in the same fashion, that is, as "the locus of a point which starts from a fixed position (called the origin) on a given straight line, and moves along the line at a uniform speed, while the line itself revolves at uniform speed about its origin." (5)

In subsequent studies on the question of tangents, Descartes, Torricelli (Viviani), John Wallis, Barrow and Newton will all "borrow," to one degree or another, from the original discoveries of Roberval and Fermat. Some will mention whom they are borrowing from, others will not. It is worth noting that there is a strong tendency for plagiarism among the British school, and especially since it has been under the influence of Venice. It is interesting to note that the method one chooses (or prefers) in determining tangents will reveal how much one is attached to linearity or, non-linearity. From this vantage point, the Roberval method of {**COMPOSITION OF TWO MOTIONS}** ***is the most natural method, and will be the immediate precursor to the transcendental {*****MEASURE OF NON-LINEARITY}** ***that Leibniz will introduce with the method of {*****OSCULATION}.**

Huygens developed a somewhat different approach that had a seminal influence on Leibniz, as have both Roberval, and Pascal. In the year 1674-75, Leibniz studied Roberval and Pascal's work, and developed a method of establishing linear tangents that he called his method of differences [differential calculus]. It is expressed by his famous formula dx/dy with the celebrated {**CHARACTERISTIC TRIANGLE}** ***that Leibniz derived from the first three propositions of Pascal's {*****TRAITE DES SINES DU QUART DE CERCLE},** ***which are, themselves, derived from Roberval's propositions 4 and 32 of his study of the {*****RATIO OF THE ARC OF A CIRCLE TO ITS DIAMETER}.** ***(6)***

In summary, Roberval has made the following significant discoveries. He has discovered a method of integration by means of the infinitesimals, and has made the first application of this calculus to the quadrature and cubature of plane and solid cycloidal figures. He is the first to have discovered the nature and the property of the cycloid, and to have found its companion curve, the sine curve, known in his days as the "Roberval curve." He found the area between the two curves, as well as the entire area under the cycloid, as well as centers of gravity, and the volumes of the principal solids of revolution that are generated by revolving their figure around an axis. His discovery of the construction of tangents by means **of {COMPOSITION OF MOTIONS} and his method of {PARALLEL TRACTION} from a {GENERATING HEREDITARY CIRCLE},** ***make Roberval the original pioneer in the domain of the differential and integral calculus that Leibniz will develop after him.***

{A SHORT BIOGRAPHY OF ROBERVAL}.

Gilles Personne was born in 1602 in the town of Soissons, from a simple family of farmers. At a fairly young age, Roberval had the opportunity of studying Greek, Latin, and the geometry of Archimedes. During the 1620's, he had the opportunity to travel throughout France, and met Pierre Fermat at Bordeaux. In 1628, he established himself in Paris where he met Father Mersenne, and Etienne Pascal, the father of Blaise. Mersenne, Etienne Pascal, Roberval, and others were holding regular geometry seminars twice a week, at the house of Pascal. It was during one of those seminars that Mersenne introduced the {**PARADOX OF THE CYCLOID}** ***that had baffled Aristotle and every geometer since antiquity. According to Blaise Pascal, Mersenne had been pondering over this problem since 1615. Considering the problem too difficult, Roberval first let problem incubate, and mature for six years, a period during which Roberval plunged himself into an intense study of Archimedes.***

In 1634, Roberval made a crucial discovery of principle that led him to easily solve the cycloid problem that neither Galileo, nor Descartes had been able to do. He discovered the beautiful and natural method of {**THE HEREDITARY GENERATIVE PRINCIPLE OF THE CIRCLE}** ***to which he applied infinitesimals. Because of the original works that he did on Archimedes, he won the chair of Ramus at the College Royal de France where he remained as the geometry teacher for 41 years, until his death in 1675. Since the chair could only be maintained by winning the geometry contest, Roberval kept most of his discoveries secret, and unpublished especially the geometry of the cycloid.***

At the beginning of 1638, Descartes launched a major attack against the book of Pierre Fermat, {**DE MAXIMIS ET MINIMIS},** ***which included an application of the method of tangents to curves. Roberval and Etienne Pascal took the defense of Fermat against Descartes. This made a lot of noise all over Paris, until 1640 when Roberval and Descartes began to confront each other, directly, on several scientific matters, among others, the {*****PARADOX OF THE CYCLOID}.**

Descartes was so afraid to confront Roberval publicly on the question of the cycloid that he declined to debate him, and chose instead to consider that transcendental curves should be excluded altogether from geometry. Descartes believed that if you ignore their existence, they go away. Mersenne wrote to Descartes on April 28, 1628, giving him a report "on a number of beautiful and new geometrical and mechanical speculations" that Roberval had made. Descartes pondered on the cycloid question for a whole month, and replied back to Mersenne, on May, 27, 1638, in an obvious fit of rage: "I cannot see why there should be so much noise about the fact of finding such an easy thing which anyone with the least knowledge about geometry cannot miss finding provided he is looking for it"...[and knowing that Father Mersenne would show the letter to Roberval, Descartes added this provocative sarcasm] "if I were to praise myself for having found such a thing, it would seem to me that it would be the equivalent of someone who looks inside of an apple which he has just cut in half and would be bragging about having discovered something that no one else but he had ever seen."

*Why would the famous Geometer Descartes go into such an infantile fit about Roberval's discovery of the cycloid? What is it that was so threatening in such a little discovery that would get the great Descartes so upset? Ego competition is one thing, yes, but there is more. There is a certain degree of emotion that is attached to a new {***DISCOVERY OF PRINCIPLE***}, and this is the kind of emotion that Descartes, like Kant, was incapable of expressing, because they both denied the fundamental emotion linked to creativity; that is, {***AGAPE***}. So, Descartes put himself into an infantile fit because he denied in himself the simplicity of a beautiful natural geometric construction that put into question his entire worldview, which he refused to change. For Descartes the cycloid is a real anomaly, and by ignoring its existence, he hoped that it would go away. He simply refused to open his heart and mind, as a child would normally do, in awe before the beauty of solving the paradox of a new discovery.*

A useful comparison should be made here with respect to the attitude of Pierre Fermat, vis a vis the same discovery of Roberval. In a letter to Mersenne, in July, 1638, Fermat writes that Roberval had miscalculated the area of the cycloid, and that his "proposition was false." So, as in the case of Descartes, Fermat also failed to see the importance of the discovery. However, after further examination, Fermat honorably amended himself by writing again to Mersenne, a few days later, on the July 27, 1638, in which he added: "I am taking the pen in order to justify M. Roberval against the too quick censure that I have made about his proposition for the cycloid, and I am refuting myself even before the refutation comes from him." (7) Contrary to Descartes, Fermat had taken the time to reflect on the correctness of Roberval's construction.

In 1644, Roberval published a treatise on the Astronomy of Aristarchus of Samos, a Treatise on Mechanics in 1645, a Treatise on the Parabola in 1651, and in 1666, he became one of the seven scientists to create the first Royal Academy of Sciences in Paris.

In 1658, Pascal launched a beautiful challenge to all of the geometers of Europe. In June and October of that year, with the agreement of Roberval, Pascal launched, under the pen name of Amos Dettonville, an international contest to investigate the "secrets" of the construction of the cycloid. The contest was not only aimed at stirring the geometric curiosity of his time, but also, had the higher purpose of provoking, among the European Intelligentia, the spirit of investigation into an entire new area of transcendental curves which will result in the discovery *of the calculus. This extraordinary account is found in Pascal's {*HISTORY OF THE CYCLOID} *by A. Dettonville that Irene Beaudry will make public soon.*

NOTES

*(1) Leon Auger, {*GILLES PERSONNE DE ROBERVAL}*, Paris, Librairie Scientifique A. Blanchard, 1962, p.42.*

*(2) {*DE TROCHOIDE EJUSQUE SPATIO, MSS. FDS. LAT}. 2340 (B.N.) *Paris*

*(3) Evelyn Walker, {*A STUDY OF THE TRAITE DES INDIVISIBLES OF GILLES PERSONNE DE ROBERVAL}*, Teacher's College, Columbia University, N.Y 1932, p.176-177]*

(4) Evelyn walker, Op. Cit., p.132.]

*(5) Sir Thomas Heath, {*ARCHIMEDES}*, p.42. Walker adds that Roberval was not the first to observe the separation of motion of a point into two components, and that, in this case, according to Paul Duhem, Roberval was influenced by Leonardo da Vinci through the works of Baldi, in relationship to the studies of the "composition of concurring forces." [See Paul Duheim, {*ETUDES SUR LEONARD DE VINCI}, 2 Vol., Paris, 1906-1913, and {LEONARD DE VINCI ET LA COMPOSITION DES FORCES CONCOURANTES}, *Bibl. Math., 43, 1903. Pp.338-343]*

(6) Evelyn Walker, {Op. Cit.,} *p.178 and 219.*

(7) P. Tannery et Ch. Henry, {OEUVRES DE FERMAT}, *Supplément, p. 87-93.*
The most significant contribution of Fermat to the efforts of developing a {UNIVERSAL PRINCIPLE OF ISOCHRONISM} *occurred in 1660's when he discovered a method of minimum for the propagation of light refraction. In May 1662, Fermat received a letter from the Cartesian Clerselier, in which, he was told that his* {PINCIPLE OF LEAST TIME} *was at best acceptable as a "moral principle" but was unacceptable in physics. This unleashed a total war between the Cartesians and Fermat. Clerselier wrote:*

"Sir,

Do not think that I am answering you today because you think you have obtained the objective of troubling the peace of the Cartesians...

"1. The principle that you consider as the foundation of your demonstration, that is, that nature always acts along the shortest and simplest pathways, is nothing but a moral principle and not at all physical, that is, not and could not be the cause of any effect of nature….

"It is not the case, because it is not this principle that makes nature act, but rather, the secret force and the virtue that is in every thing, that is never determined by such or such an effect of this principle, but by the force that is in all causes that come together into one single action, and by the disposition that is actually found in all bodies upon which this force acts.

"And it could not be otherwise, or else, we would presume nature to have knowledge: and here, by nature, we mean only this order and this law established in the world as it is, which acts without foreknowledge, without choice and by a necessary determination.

"2. This same principle must put nature in an unresolved state, not knowing how to determine itself, when she has to pass a ray of light from a light medium through to a denser one. Because, I ask you, if it is true that nature must always act by the shortest and simplest pathways, since the straight line is undoubtedly both the shortest and the simplest of all, when a ray of light has to travel from a point of a light medium and end in a point of dense medium, isn't it the case that nature must hesitate, if you wish her to act by the principle of following a straight line soon after a break, since, if the latter is the shortest in time, the other is shorter and simpler in measure? Who will decide and who will pronounce himself on this matter?

"3. Since time is not what moves things it cannot either be that which determines movement, and once a body is moved and is determined to go in some direction, there is no apparent reason to believe that the time, more or less short, would force this body to change its determination, that which does not act and which has no power over it. But, since all speed and all determination of the movement of a body depends on the force and the disposition of that force, it is quite natural, and this is my belief, that it is better physics to say, as Mr. Descartes says, that the speed and determination of a body change because of the change occurring within the force and within the disposition of that force which are the real causes of its movement, and not to say, like you do, that they are changed by a design that nature has to always proceed by the pathway of least time, a design which she cannot have because she is unknowing and which cannot have any effect on the body."

Thus, Clerselier focused the debate from the false* {AXIOMATIC ASSUMPTION} *known as Cartesian "dualism": the separation of human reason from the real world. As a result, it becomes inconceivable that the choice of a least action pathway of light, as opposed to any other, could be the result of an* {INTENTION}, *or what Leibniz properly attributed to a* {FINAL CAUSE}. *In fact, the ray of light does not hesitate and deliberate before deciding on the shortest path. It does not have that choice. The universe is* {HARMONICALLY ORDERED} *in such a way that the very* {INTENTION OF THE LEAST PATHWAY}, *is built into the behavior of the ray of light by God, as a* {PRE-ESTABLISHED PRINCIPLE OF HARMONY} *within the universe as a whole.

The same objection to Fermat's method still continues today against Bernoulli's* {BRACHISTOCHRONE}. *In their textbook on mathematics,* {WHAT IS MATHEMATICS?} *Richard Courant and Herbert Robbins wrote: "Bernoulli's 'proof' is a typical example of ingenious and valuable mathematical reasoning which, at the same time, is not at all rigorous. There are several tacit assumptions in the argument, and their justification would be more complicated and lengthy than the argument itself.... The question as to the intrinsic value of heuristic considerations of this type certainly deserves discussion, but would lead us too far astray."

*****

APPENDIX - 2

A LEIBNIZ - LAROUCHE METHOD OF GENERATING THE CATENARY-TRACTRIX.

by Pierre Beaudry

In a memorandum on the {**CATENARY FUNCTION}*****, written in 1989, Lyndon H. LaRouche established a crucial difference between*** **{FORMAL CONSTRUCTIVE GEOMETRY}** ***and*** **{PHYSICAL GEOMETRY};** ***that is, a crucial difference between what physical nature is capable of accomplishing in the real physical universe, and what geometry is capable of offering as a formal representation of the real universe. As we shall see,*** **{THERE IS ALWAYS MORE IN WHAT NATURE OFFERS THAN CAN BE MEASURED BY OUR GEOMETRY},** ***and for that reason, even when we cannot apply a formal geometry to the real world, we must always look for the "less inadequate geometric pathway", which does not mean it is the easiest. In all events, the*** **{LESS INADEQUATE GEOMETRIC PATHWAY}** ***must always be the simplest that can be drawn from the shadows of Plato's cave.***

The following pedagogical exercise is aimed at showing how this difference can be made cognitive with the construction of the **{CATENARY-TRACTRIX}.** ***I will first reproduce the entire LaRouche 1989 statement, and secondly, I shall proceed by applying the LaRouche principle to our construction. LaRouche wrote:***

"{REASONING FROM ABSTRACT CONSTRUCTIVE GEOMETRY, IT IS THE CYCLOIDS AS SUCH WHICH DEFINE ISOCHRONISM. IN REAL PHYSICS, IT IS THE CATENARY AS THE EVOLUTE THAT CORRESPONDS TO THE PHENOMENA ASSOCIATED WITH "FORCE" AND "ACTION AT A DISTANCE" IN THE TERMS OF REFERENCE OF A DISCRETE MANIFOLD. IT IS IMPORTANT THAT THE SIGNIFICANCE OF THIS POINT NOT BE OVERLOOKED.

"THE FEATURE OF THE CATENARY JUST REFERENCED, IS AN ANOMALY FROM THE STANDPOINT OF FORMAL CONSTRUCTIVE GEOMETRY. THUS, IT DEFINES AN INADEQUACY OF THAT GEOMETRY AS A REPRESENTATION OF THE PHYSICAL WORLD. YET, IT SHOWS THAT THE ANOMALY IS RESOLVED, BY A MORE ADEQUATE GEOMETRY THAT TAKES THE GENERATION OF THE CATENARY INTO ACCOUNT.

"IN THIS SENSE, WE MUST TREAT THE CATENARY AS BELONGING TO THE EXTENDED FAMILY OF CYCLOIDS. MOREOVER, WE MUST PLAY THAT BACK INTO OUR VIEW OF PHYSICAL GEOMETRY AS A WHOLE. THAT IS TO SAY, THAT THE GEOMETRY WHICH DEFINES AS NECESSARY, FROM A GEOMETRIC

STANDPOINT, THE ELEMENTARY CHARACTERISTICS OF PHYSICAL ACTION, IS, BY DEFINITION, THE LESS INADEQUATE PHYSICAL GEOMETRY.

"HENCE, THIS CORRECTION MUST BE REFLECTED IN UNDERSTANDING THE DOUBLE-CONNECTEDNESS OF ISOPERIMETRIC AND ISOCHRONIC ACTION IN DEFINING PHYSICAL LEAST ACTION ELEMENTARILY.

"THE TRAP TO BE AVOIDED, IN THIS UNDERTAKING, IS THE TENDENCY TO FALL BACK INTO THE REDUCTIONIST'S VIEW THAT IT IS GRAVITATION, FOR EXAMPLE, WHICH SHAPES THE CURVATURE OF PHYSICAL SPACE-TIME. SINCE ISOCHRONISM IS A NECESSARY PRE-CONDITION FOR THE VERY NOTION OF UNIVERSAL LAW, THE ISOCHRONISM ITSELF IS NOT A FUNCTION OF GRAVITY'S ACTION IN SHAPING THE CURVATURE OF SPACE, BUT, RATHER, GRAVITY IS CAUSED BY ISOCHRONISM."} [89-04-5/LAR004]

In a formal way, it was Leibniz, his followers and associates, such as Huygens, Fermat, Roemer, and John Bernoulli, who made the crucial discoveries of principle pertaining to **{ISOCHRONISM}** ***from the standpoint of*** **{ABSTRACT CONSTRUCTIVE GEOMETRY},** ***as is applies to cycloids. The curve of*** **{MOST RAPID DESCENT}** ***that became known as Bernoulli's*** **{BRACHISTOCHRONE}** ***is, geometrically speaking, the*** **{ISOCHRONIC CURVE}** ***par excellence, that is, the ordinary cycloid pathway along which any number of rolling balls will arrive at the lowest point of the curve, at the same time. Other types of cycloids reflect this characteristic*** **{ISOCHRONISM}** ***as well.***

However, from the standpoint of physics, (which is not universally characterized by rolling balls), the **{CATENARY EVOLUTE}** ***is the physical expression that most adequately represents universal law whereby the*** **{FORCE OF LEAST ACTION}** ***is distributed throughout the universe; that is,*** **{THE MOST WELL ORDERED AND DISTRIBUTED FORCE}** ***in the universe as a whole, precisely because the universe in which we live "holds together, as Kaestner put it, "the spider's thread with the same force that pushes or pulls the planets around the sun." In this fashion, the*** **{CATENARY EVOLUTE}** ***represents the least inadequate expression of least action in the physical world, making it*** **{THE BEST OF ALL POSSIBLE WORLDS}.**

This means that **{ACTION AT A DISTANCE}** ***simply does not exist in the form prescribed by so-called*** **{BRITISH SCIENCE}.** ***That is to say, the push-me, pull-me sort of mechanical causality simply does not reflect the lawful ordering of the universe. It is unfortunate that most physicists today have accepted this silly British world view whereby the world we live in is represented as a bunch of hard balls hitting one another, more or less randomly, as if in some sort of mad statistical billiard game with no other ordering principle than a*** **{MYSTERIOUS FORCE}** ***at a distance, which determines the*** **{MEASURE OF GRAVITY}** ***in the universe according to the*** **{INVERSE SQUARE LAW OF THE DISTANCE}.**

The anomaly that no abstract geometry, nor any **{CYCLOID}** ***can account for, but which is reflected in the*** **{CATENARY EVOLUTE},** ***is expressed by the fact that any small change in its curvature changes the catenary curve as a whole. That is to say, any minute change in the most remote corner of the universe, as small as an unlawful construction of a spider's web,***

would cause the universe to change as a whole, such that the entire world would become a different universe. The slightest change means a complete change. This is the reason why the current unlawful Greenspin manipulations of the monetary system can cause the entire biosphere of this planet to disintegrate. In case anyone wondered, this is also the reason why spiders cannot knit sweaters, and bees cannot produce strawberry jam. This kind of anomaly cannot be represented by a formal geometry, but the **{PROPORTIONATELY DISTRIBUTED FORCE}** ***along the*** **{CATENARY EVOLUTE}** ***does represent the anomaly in the least inadequate way, that is, as well as can be represented by the shadows on the dimly lit wall of Plato's Cave.***

The point that LaRouche is making here is that there is only one thing wrong with formal Geometry. It cannot provide an adequate representation of the real world. It is crucial to understand, as LaRouche points out, that all forms of **{GEOMETRY}** ***are inadequate. For a long time,*** **{EUCLIDEAN GEOMETRY}** ***was thought to be an appropriate representation of the world we live in. It is not.*** **{EUCLIDEAN GEOMETRY}** ***merely represents an abstraction of the linear impressions that the outside world projects on our sense perception.***

What LaRouche requires of us here is that our treatment of the **{CATENARY}** ***must be such that its characteristic least action reflects the*** **{DOUBLE-CONNECTEDNESS OF ISOPERIMETRIC AND ISOCHRONIC ACTION}.** ***How can this be demonstrated? Can this be demonstrated by*** **{FORMAL GEOMETRY}?** ***If so, how can the*** **{CATENARY}, as** ***a member of the same extended family of curves as the cycloids, represent that it is the*** **{LESS INADEQUATE PHYSICAL GEOMETRY}** ***to reflect the singular discontinuity of passing from*** **{ISOPERIMETRIC POSITIVE CURVATURE TO ISOCHRONIC NEGATIVE CURVATURE}?** ***If that is the question that LaRouche wants us to answer, then, we believe that part of the answer can be found by making use of the Leibnizian construction of the*** **{CATENARY-TRACTRIX}** ***by means of his method of*** **{INVERSION OF TANGENTS}.**

This Leibnizian method shows us how (Figure 23), by **{PARALLEL TRACTION AND INVERSION},** ***to transfer the radius*** **{AE}** ***and the tangent*** **{DE}** ***of the quarter circle*** **{ACEF}** ***and transform then into the tangents*** **{(D)(E)}** ***and*** **{(A)(E)}** ***of the catenary and the tractrix respectively, that is, from the isoperimetric domain of positive curvature to the isochronic domain of negative curvature. Thus, the*** **{CATENARY-TRACTRIX}** ***is defined as nothing but the*** **{EVOLUTE-INVOLUTE}** ***curves generated by the*** **{INVERSION OF TANGENTS}** ***of a quarter circle. Note furthermore that a third curve*** **{F(B)G},** ***the*** **{SINE CURVE},** ***is also created by the same process. Note that the area of*** **{ISOCHRONIC NEGATIVE CURVATURE}** ***between the*** **{SINE CURVE}** ***and the*** **{TRACTRIX}** ***is obtained by the Roberval-Fermat construction of the cycloid, as in Figures. 17,18,19, and 22.***

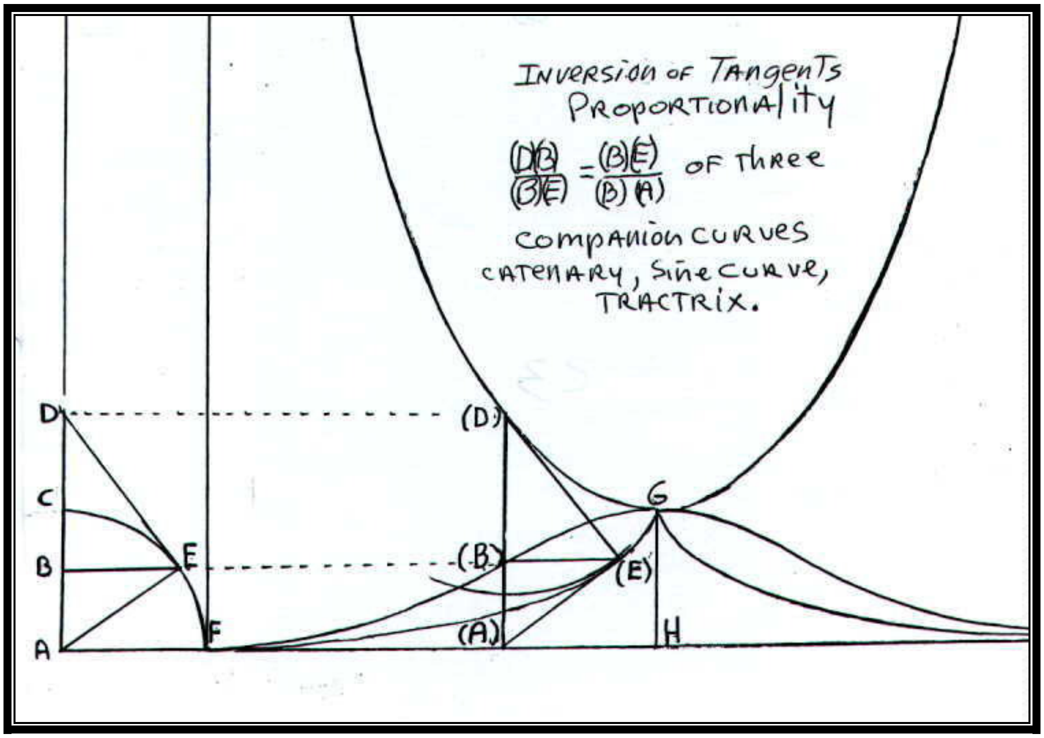

Figure 23. Doubly connected tangents to the isoperimetric circle and to the isochronic catenary-tractrix.

Lastly, LaRouche warned us about the trap of reducing the curvature of physical space-time to* {GRAVITATION}. *It is* {ISOCHRONISM} *that determines* {GRAVITATION}, *and not the other way around. And, I might add, it is the Kepler harmonically ordered* {PROPORTIONALITY OF ISOPERIMETRY AND ISOCHRONISM} *which establishes the bridge, so-to speak, between* {ABSTRACT GEOMETRY AND PHYSICAL GEOMETRY}. *Since there is always more in what nature offers than can be measured by our geometry, there can never be an* {EQUATION} *between the two, only an* {INCOMMENSURABLE PROPORTION}. *Thus, the* {LESS INADEQUATE} *situation that is created is overcome and the anomaly gets resolved when the physical reality and the mental process relate in a manner such that* {THIS IS TO THIS AS THAT IS TO THAT}: *The incommensurable proportionality between* {ISOPERIMETRIC} and {ISOCHRONIC}. *As Figure 23 shows:

$$\frac{DB}{BE} :: \frac{BE}{BA} \quad \bullet\bullet \quad \frac{(D)(B)}{(B)(E)} :: \frac{(B)(E)}{(B)(A)}$$

This leads us to demonstrate how the very construction of the* {CATENARY-TRACTRIX} *depends on such {*PROPORTIONALITY}, *by the very action of constructing it.

***That is to say, you discover how to walk by walking. The curves are constructible precisely because the reason is given through the proportionality of such functions, within* {INTERVALS OF INTERVALS}. *It is therefore by causing this inadequacy to occur, by means of the Leibniz* {PRINCIPLES OF OSCULATION, TRACTION, AND INVERSION OF TANGENTS}, *that a higher dimensionality of least time and least pathway offers a Platonic solution to the problem. In other words, the following* {FORMAL GEOMETRIC METHOD} *must walk through the very steps of the discovery of principle that was developed in the* {MENO DIALOGUE} *of Plato; that is, the successive three steps of an axiomatic change by* {PERPLEXITY- DISCOVERY-SOCIALIZATION}, *otherwise known, in Rabelaisian language, as* {PERPLEXITY-WONDER-LAUGHTER}.**

{HOW TO TURN THE CONSTRUCTION OF THE CATENARY AND TRACTRIX CURVES INTO CHILD'S PLAY!}

The following construction of the catenary and tractrix curves is based on Gottfried Leibniz's method of {*DETERMINING A CURVE FROM A PROPERTY OF ITS TANGENTS}, *and following Lyndon LaRouche's method of discovering by* {UNDERLYING MULTIPLY-CONNECTED CIRCULAR ACTION.} *This construction is entirely geometrical and does not require a physical chain or any mathematical formula.

{THEOREM:}

{GIVEN TWO FAMILIES OF CONCENTRIC CIRCLES, GT AND GH, GIVEN TWO PARALLEL LINES G AND H, AND GIVEN TWO TANGENTS TC AND GT, FIND THE CATENARY CURVE GOING THROUGH POINT C AND THE TRACTRIX CURVE GOING THROUGH POINT T, USING ONLY THE PROPERTY OF THEIR TANGENTS.}

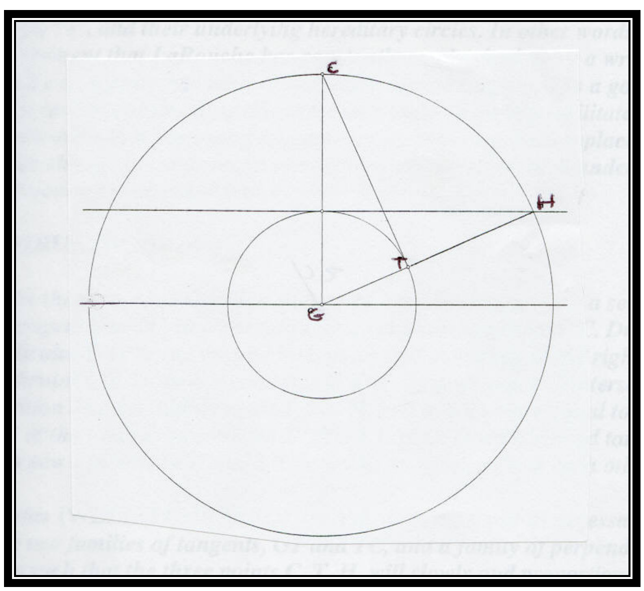

Figure 24. Theorem of the catenary-tractrix

First and foremost, in order to construct this theorem, the reader must shed the **{FALSE UNDERLYING EUCLIDEAN ASSUMPTION}** *which claims that a tangent always has to be found by means of the curve to which it is tangent. In the present case, this is not true. Here, it is the inverse that holds true - that to find the curve, it is only required to use the universal property of its tangents. In other words,* **{THE CART MUST BE PUT IN FRONT OF THE HORSE!}** *However absurd this may appear to be, such is the method of inversion that Leibniz has applied to several of his discoveries published in* **{ACTA ERUDITORUM},** *and which follow his provocative axiom busting proposition*: **{CONSTRUCT A CURVE FROM A GIVEN PROPERTY OF ITS TANGENTS}.** *(1)*

Secondly, the following construction simply requires the use of a straight edge (preferably a right angle triangle), which must be handled with extreme precision, requiring a steady hand, patience, and a keen sense of fine-tuning. This signifies that this construction does not actually require the explicit use of a compass or of a circle. The two families of circles, shown here in Figure 25, are aimed at emphasizing the **{BOUNDARY CONDITION}** *between the catenary and tractrix curves, and their underlying hereditary circles. In other words, following the requirement that LaRouche has constantly emphasized, once a wrong axiomatic assumption has been rooted out, it must be replaced by a good one. Thus, the task of constructing this theorem should be greatly facilitated if the reader discovers that the wrong tangency assumption has been replaced by an axiomatic change in the boundary conditions expressed by such underlying multiply connected circular action.*

{CONSTRUCTION}:

Given the theorem represented in Figure 24, start by determining a second point C' on the tangent line TC, in a position located slightly under point C. Drop a second perpendicular C'G' to the ground line, in a position slightly to the right of the first perpendicular CG. Draw a second line G'T'H' through a new point T', intersecting line H, in a position located slightly to the left of H, and at a distance equal to C'G'. Mark point T' at the same distance from G' that T is to G. Draw a second tangent T'C'. The two new tangents G'T' and T'C' must be at right angle to each other.

Repeat this **{WELL-TEMPERED}** *process, as many times as necessary, to produce two families of tangents, GT and TC, and a family of perpendiculars CG, in a way such that the three points C, T, H, will slowly and proportionately converge toward each other, and shall eventually coincide somewhere between the two families of circles on line H.*

In determining the series of points C, C', C'', C'''…etc., T, T', T'', T'''…, etc., and H, H', H'', H'''…etc., make all of the tangent points C C's… move from left to right in a downward curve, forming the catenary curve; make all of the tangent points T T's…move from left to right in an upward curve, forming the tractrix curve; and make all of the points H H's… move from right to left, along the straight line H. All of the points C, T, H will converge toward each other into a singular region of discontinuity that Leibniz identified as a caustic of **"{CONCURRENT LINES}".** *This reflects the same determined ordering which underlies the optical concurrence of rays of light inside of a cylindrical mirror, a high density of singularities into an* **{EDGE OF CAUSTIC INVERSION}.** *(2)*

Moreover, this is the way to determine the locus of an **{EVOLUTE CURVE}** ***generating an*** **{INVOLUTE CURVE},** ***through an*** **{EDGE OF EVOLUTE-INVOLUTE INVERSION}.** ***Both curves are defined by the apportioning of tangents TC, and GT, all of which indicate that, even though the drawing may show a finite distance between all of those points, one must understand that all of the points C, T, and their respective tangents, are so infinitesimally close to each other, that you could not conceivably put a third point, or a third line, between them. The curvature between such points must be thought of as being so fine that all of the tangents, which Leibniz calls*** **{DIFFERENTIAL CHARACTERISTICS},** ***should not even be conceived of as being made up of separate finite lines, but, rather as an integral curving surface envelope of infinitesimally small differentials between the two curves which generally appears, in its natural form, as a*** **{CAUSTIC ENVELOPE OF LIGHT}.**

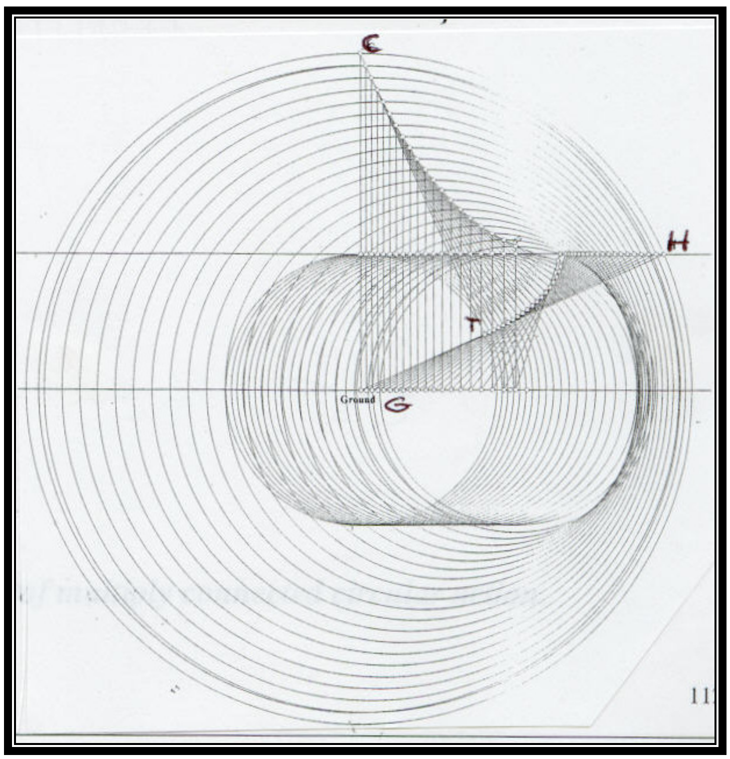

Figure 25. Computer simulation of multiply connected circular action.

Figure 25 shows a computerized version made by a young student, Jacob Welsh, (3) to represent a simulation of the boundary conditions of the underlying multiply connected circular action of the entire process. However, the problem that this computer program reveals is that it cannot replicate such a complex form of action truthfully. **{THE COMPUTER CANNOT REPLICATE THE LEIBNIZ CALCULUS!}** ***(4) No benchmarking formula can reproduce what only a human mind is able to accomplish. That is why, as the reader can appreciate in Figure 26, the computer encounters*** **{A NON-LINEAR SINGULARITY, A DISCONTINUITY,}** ***at a paradoxical point when it reaches the converging region of CTH. At that point, the computer program is forced to stop and break down, because the linear axiomatic***

assumptions built in its program are not valid. In other words, this entire computer drawing is a fallacy of composition

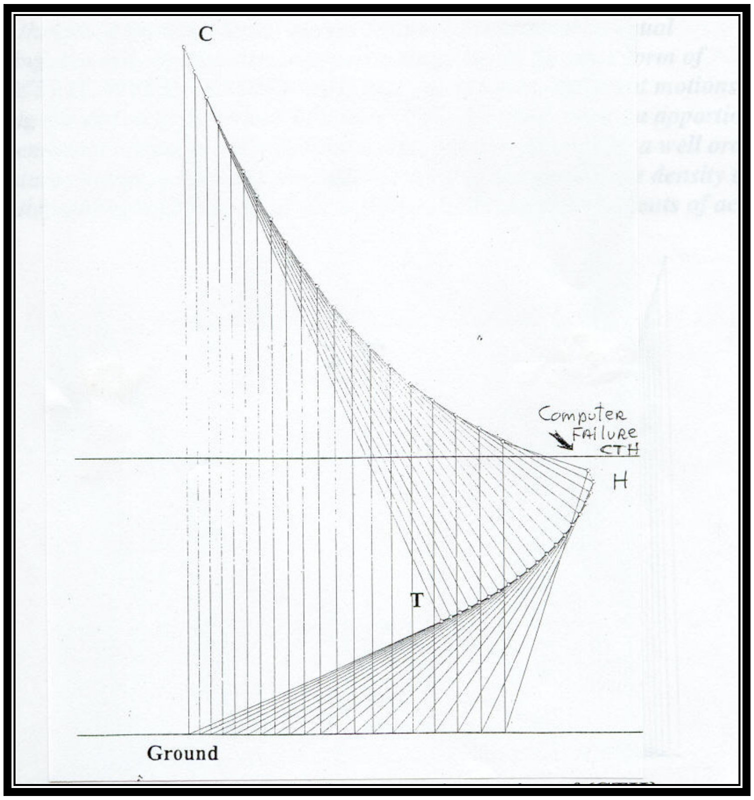

Figure 26. Computer breakdown in the converging region of {CTH}.

This step represents a crucial moment of* {PERPLEXITY}. *This is the beginning of an axiomatic change which demonstrates what Lyndon LaRouche has been arguing all along, against Norbert Wiener and John Von Neumann's Cybernetics, that the paradox of the regular non-uniform motion of a transcendental curve, such as the catenary/tractrix, cannot be reduced to a priori linear equations.* {BENCHMARKING DOES NOT WORK}. *The computer cannot carry out the human requirement of circular motion and non-linear cognition, any more than human cognition could be reduced to a series of equal linear segments.

The only way to solve the paradox that the computer cannot solve is to construct the theorem cognitively, and physically by hand; that is, by assuming, yourself, that such transcendental curves cannot be measured by equal partitioning, but only by non-linear apportioning, that is, by some form of* {GEOMETRIC WELL-TEMPERING} *between the three different motions underlying the different directions of points* {C, T, H}. *And, such an apportioning of transcendental curves is not only non-linear, but it is affected by a well ordered non-constant change, which has the added feature of increasing the density of singularities within each and all of the infinitesimally small increments of action.

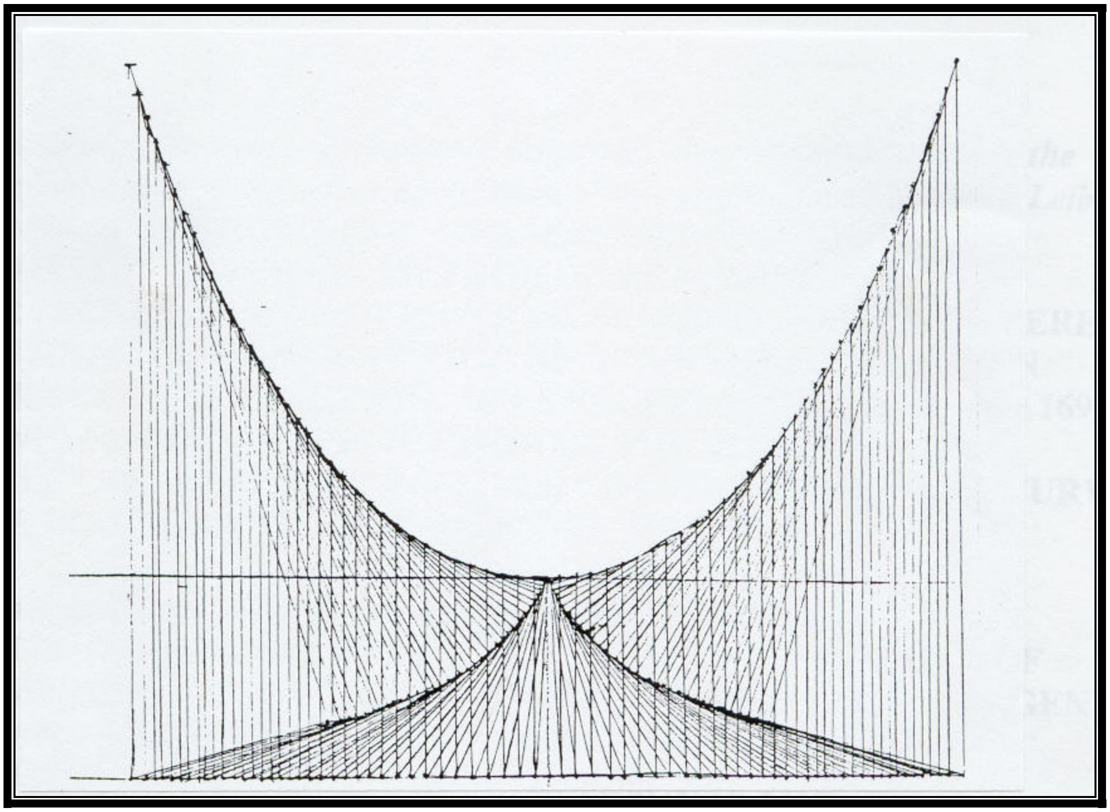

Figure 27. Jacob's hand drawing of the catenary-tractrix.

As the reader is now capable of recognizing, this completed construction of the **{CATENARY-TRACTRIX}** ***required this kind of closure. That is the moment of*** **{DISCOVERY}.** ***No problem could ever be solved without establishing such boundary conditions. It is the bounded multiply connected determination of the*** **{CATENARY EVOLUTE}** ***that made the solution of the problem possible. Unless boundary conditions are set, the mind becomes a wanderer on an ocean without a shore. It was the knowledge of the boundary conditions that told Columbus where to land.***

Thus, in constructing this {**CATENARY-TRACTRIX}*****, by hand, the reader will not only be happy to discover that he can relive something creative that the computer cannot do, but he, or she, will truly enjoy sharing the crucial discovery of the difference between what the linear computer cannot do, and what only a human being can accomplish with the cognitive geometric methods of Leibniz and LaRouche. (5) You don't have to take my word for it, you can prove it to yourself:*** **{BELIEVE NOTHING THAT FOR WHICH YOU CANNOT GIVE YOURSELF A CONSTRUCTIVE PROOF}.** ***By replicating, and*** **{SOCIALIZING}** ***this experiment with others, you will have brought closure to the scientific process of a creative discovery of principle. Enjoy it, and pass it on!***

NOTES

(1) We are given the property whereby a tangent is always at right angle to the normal of the curve. For further application of this question, see Gottfried Leibniz, **{EXTENSION OF GEOMETRIC MEASUREMENTS USING AN ABSOLUTELY UNIVERSAL METHOD OF REALIZING ALL QUADRATURES BY WAY OF MOTION: ACCOMPANIED BY DIFFERENT PROCEDURES OF CONSTRUCTION OF A CURVE FROM A GIVEN PROPERTY OF ITS TANGENTS, ACTA ERUDITORUM}, September 1693,** *and* **{A NEW MODE OF APPLICATION OF THE DIFFERENTIAL CALCULUS TO DIFFERENT POSSIBLE CONSTRUCTIONS OF A CURVE FROM A PROPERTY OF ITS TANGENTS}, JULY, 1694***,*

(2) Leibniz developed these questions extensively, especially his **{CONSTRUCTION OF A CURVE DERIVED FROM AN INFINITY OF LINES WHICH ARE WELL ORDERED, CONCURRENT, AND TANGENT TO IT; AND A NEW APPLICATION FOR THE ANALYSIS OF INFINITIES,} ACTA ERUDITORUM}, Leipzig, April 1692***.*

(3) Jacob Welsh was 11 years old when he constructed these computer and hand models for the catenary/tractrix, on April 8, 2001. How old are you?

*(4) Thus, the construction of this catenary-tractrix {***ENVELOPMENT-DEVELOPMENT}** *function required the application of three new discoveries of principle developed by Leibniz: 1) the {***PRINCIPLE OF TRACTION MOTION}** *of the tractrix involute, 2) the {***OSCULATING PRINCIPLE}** *of the catenary evolute, 3) the {***PRINCIPLE OF INVERSION OF TANGENTS}***, all from the hereditary circle. These three principles are harmonically conjugated according to the arithmetic-geometric mean proportionality that Leibniz had initially discovered in the relationship of the catenary curve and the logarithmic curve. See* **{FIDELIO} G.W. Leibniz, {TWO PAPERS ON THE CATENARY CURVE AND LOGARITHMIC CURVE}, Spring 2001.**

(5) Next, the reader must apply the Riemannean requirement for the strong case of application to the real physical universe. When this geometric principle of construction is applied to the analysis situs of a real organizing situation; that is, within the reality of an ongoing historical change in the world strategic situation, as in the case of the current world financial breakdown, such a pathway of isochronic least time becomes a metaphor representing the least action of changing the world view of another person, in the least time; that is, by transforming what appears to be, in the subject's mind, a high density of singularities per small area of action into a new world view.

For more information about constructive geometry, write to: pierrebeaudry@larouchepub.com

FIN

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