# PARMENTIER'S FOOTNOTES AND COMMENTARIES ON THE LEIBNIZ METHOD OF QUADRATURES AND OF INVERSION OF TANGENTS.

(Requested by Michael Kirsch for the Basement crew, July 26, 2008.)

by Pierre Beaudry, 7/28/2008.

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G. W. Leibniz, {*LA NAISSANCE DU CALCUL DIFFERENTIEL*}, Introduction, translation and notes by Marc Parmentier, Paris, Vrin, 1989, Chapter XV, pp.247-267.

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## XV

## SUPPLEMENTUM GEOMETRIAE DIMENSORIAE SEU GENERALISSIMA OMNIUM TETRAGONISMORUM EFFECTIO PER MOTUM: SIMILITERQUE MULTIPLEX CONSTRUCTIO LINEAE EX DATA TANGENTIUM CONDITIONE[1]

*This treatise on the construction of a* geometric machine *cannot but provoke in the reader a profound state of perplexity. The errors of calculation, the mistakes of precipitation call for indulgence, but what are we to think of the treasures of the imagination deployed by Leibniz in order to set up a sophisticated mechanism for the purpose of calculating quadratures? Is there not in that an excess of naivety or even a flagrant use of contradictio adjecto? This challenge is throwing us off because we have reconnected with the Greek idea of a difference of nature, which was that of the Greek mathematicians, between theoretical geometry and practical geometry, a separation that the Leibnizian inventions are contradicting[2]: his taste for geometry had very early on turned toward Mixed Mathematics and even before he indulged in them he was already famous for his arithmetic machine including his machine for generating quadratures, the* integraph, *that he introduces as the fruit of a very old meditation, appears as the geometrical unresolved case[3]. First of all, in accordance with the spirit of his time, he*

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[1] Extension of geometric measurements using an absolutely universal method of realizing all quadratures by way of motion: accompanied by different procedures of construction of a curve from a given property of its tangents. (A. E. September 1693, M. S. V p. 294-301).

[2] We have further investigated this separation with the discovery of non-Euclidean geometries.

[3] This idea is defended by M.A.B. Chtykan, "Intergrirouiouchtchi mekhanizm Leibnitsa", in Ousp. Matem. Naouk vii, 1/47, 1952. Cf. A. P Youschevitch, "Comparaisons des conceptions de Leibniz et de Newton", S. L. Suppl. Xxvii, 1978; L. von Mackensen, "Leibniz als Ahnherr der Kybernetik", S. L. Suppl xiii, 1974. We must also attribute this idea of the integraph to that of the algebraic machine invented in Paris in 1674, cf. Couturat, *La Logique de Leibniz*, p. 115, and the letters to Huygens (M.S. II, p. 15) to Oldenburg (M.S.

*finds in the physical concretization of a curve the guarantee of its geometrical nature*[4]*. But the construction of an integraph goes beyond, it is no only aimed at the concretization of curves but also at generating geometrical* ***operations****. In that sense, it is directly connected to the other contemporary attempts of Huygens and Tschirnhaus.*

*In the second part of his* Medicina mentis, *published in Amsterdam in 1687*[5]*, Tschirnhaus had proposed a method for the construction of tangents to curves that is described by a stiletto attached to a thread fixed at two given points and constrained to pass around a certain number of other points called foci*[6]*. Tschirnhaus does not give any demonstration for the generalization of his method beyond three foci and he is mistaken. His mistake is immediately found and then corrected by Fatio Duillier, which is considered a proof of his mathematical talent, as noted by P. Costabel, and will give birth to a new method*[7]*, thanks to a collaboration with Huygens. On the contrary, Leibniz does not immediately notice the mistake of his friend Tschirnhaus and lets himself be seduced by the generalization which consists in conceiving "the foci as lines". So, this generalization leads directly to the idea of the development and the generation of new curves by way of procedures that are both mechanical and abstract. Nevertheless, as he announces it in one of his first letters to l'Hospital*[8]*, he also comes up with a correction to the Tschirnhaus method, by using, as did Huygens before him, the mechanic-geometrical properties of the center of gravity of a system of points*[9]*. Even if this result were to no longer be original, it had possibly contributed in giving Leibniz the idea of a possible solution for the problem of the inversion of tangents "independently of quadratures"*[10]*, in any case it directly inspired the elaboration of his* general Rule for the composition of motions[11].

*As for Huygens, he published an article in* l'Histoire des Ouvrages des Savants *or* Journal de Rotterdam *where, after having given the rectification of the logarithmic curve proposed by Marquis de l'Hospital, he went on to construct a curve such that the portion of the tangent between a given point and the axis is a constant, a curve that he called*

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I, p. 73) and to Arnauld (P.S. I, p. 81). Finally, Couturat (Op. Cit.) reminds us that the logical process of Leibniz is constantly guided by the idea of a materialization of reasoning.

4 Cf. Supra *De linea in quam flexile*.

5 *Medicina mentis , seu Tentamen Germinae logicae in qua disseritur de methodo detegendi incognitas veritates*, Amsterdam, 1687.

6 The method is also shown in a letter of May 12, 1687, to Huygens (*Oeuvres Completes*, t. ix, p. 134-144). P. Costabel describes it completely in *Leibniz et la dynamique*, p. 60-62: "Tschirnhaus imagines a thread of constant length whose extremities are fixed at given points and whose tension is maintained by a stiletto. The idea is to study the tangent of the geometric curve that is described by the point of the stiletto…with two distinct and fixed points, one can complicate things by having the thread pass around a third fixed point…" In modern terminology, such curves are defined by a multipolar equation.

7 Cf. infra *Responsio ad Dn. Nic. Fatii Duillerii imputations*.

8 M.S. II p. 237.

9 Cf. infra *De novo usu Centri gravitatis*.

10 The paragraph of the letter to l'Hospital which mentions it follows immediately the one where Leibniz calls to mind the method of Tschirnhaus (M.S. II p. 238).

11 On which P. Costabel publishes a copy and a detailed commentary in *Leibniz et la dynamique*, p. 74-94. In reality, the call for dynamic considerations only confuses the resolution of this kinematical problem and makes its demonstration unsecured.

*Tractrix (Tractoria)*[12]*. Huygens added: " I am not submitting this curve just because of everything I have just said about it, but because there is another reason which is that it can also be described by quite a simple machine, and from there solve the problem of squaring the Hyperbola, which seems to be something worth considering on the part of geometers."*[13] *And then, he goes into the details of the traction mechanism that traces the curve. He said it is enough to pull, on a horizontal plane, "a point which by its weight, or otherwise, would cause some resistance at the end of a thread, or of an inflexible rod, and which would simply proceed from the other extremity" before concluding: "If that description were to be precise enough, in accordance with mechanical rules, and could be considered geometrical…we would have with that, along with the quadrature of the hyperbola, the perfect construction of problems that are reduced to this quadrature." Even though Leibniz only got wind of this Huygens process indirectly, because of the negligence of his librarian, and he rather had to guess at it, right away, his imagination, which is inclined towards generalizations, got a hold of the idea that made him conceive of an even more universal machine, capable of realizing other types of quadratures.*

*We have to conceive of this machine like a new* supplementum, *to be ranked on the same level as the method of series of the* Supplementum Geometricae praticae (Supplement of practical Geometry), *that is to say, like an aside capable of realizing everything that is permitted to be realized for the* completion *of Geometry. In fact, Leibniz holds such a* construction *as perfect and complete for its type. The fact that he was hoping to realize real calculations with this method is not any more surprising than the logarithmic measures that he promised himself to generate on a suspended chain, but in what manner can such constructed quadratrices bring their share to the problem of quadratures? In fact, such a solution has curiously no practical value in his eyes except theoretically. To the perplexity of Huygens, Leibniz replies: "As for my general construction of quadratures by traction, it is sufficient for science that it be exact in theory, even if it were not to be properly executed in practice. When they are composed, the majority of the most geometrical constructions are of that nature."* [14]

*The geometric value of the integraph comes from the fact that it is the theoretical idea, in other words, in the new vocabulary, exact, of a practical construction that is only possible. It doesn't matter that Leibniz might have believed or not in the concrete realization of his machine, it as for him sufficient that it were theoretically realizable, just*

[12] That is precisely the curve that Leibniz declared having discovered in Paris at the instance of doctor Perrault. The article is published in the Tome X of the Oeuvres Completes of Huygens, p. 407 in the form of a letter from "Christian Huygens to H. Basnage de Beauval".
[13] Loc. Cit., p. 409. Another note on the construction of his Tractrix also indicates to us the treasures of the technical imagination equally deployed by Huygens in his draft papers: "He imagines even a mechanism in which atmospheric pressure is involved. It is an upside down bottle, like the one he was using on the table of his pneumatic machine, and whose opening was covered with a round metal ring that was leaning against a round piece of leather or wet paper, allowing in the center the free passing of a tracing stiletto that was attached solidly to the metal ring in the middle of which it was placed. The rod that was to pull the apparatus was attached to that ring. A partial vacuum had been created inside of the bottle by heating the air inside of it, ahead of time, by means of a flame."
[14] M.S. II p.181.

*like his hope for a closure of geometry is not the one of a real achievement but only of a possible one, confirmed by the possibility of filling in the gaps of present lacunas by means of supplements. Since the term geometric is being considered from now on as synonymous with exact, we are forced to count on the talents of Leibniz, the engineer, in order to accept the exactitude of his edification plans!*

*Nevertheless, the integraph has been hit with the two natural objections that it could not avoid but raise, the first being the geometricity of its constructions, the second being the difficulty of its practical realization. The shrewdness of its inventor was to turn these two gross objections, one against the other, in order to show that they destroyed one another. The first was formulated by Jacques Bernoulli who had already addressed it against the machine that Huygens had imagined for the quadrature of the hyperbola*[15]*. More favorable, John Bernoulli later declared that he had convinced his brother*[16] *(of its correctness). In reality, after reflection, there is not more reason to frown at the geometricity of such a construction any more than at evolutes, which are also results of an unrealizable theoretical mechanism, and even for the catenary chain, whose stunning properties have seduced geometric minds. On this point, Leibniz has made a decisive intervention by specifying to Huygens that it is the tracing and not the curve that must be called the tractrix*[17]*. So, did the catenary curve not reveal the unprecedented geometric and analytical properties of a curve generated by a physical process? Definitely, everything comes down to a question of definition. If, as Leibniz would say, geometrical means exact, then such curves reveal that clearly*[18]*, and by means of the mechanical imagination, geometry must move bravely ahead of the barely outlined analysis of transcendentals. However, in this case, any reproach to the difficulty, or even to the impossibility, of applying concretely such mechanisms is beside the point. In any event, it is quite improbable that Leibniz would have really attempted its realization.*

*But, why bother with these false trails and encumbering himself, as if he was really thinking of its practical modalities, and overwhelming himself with the luxury of details, forging ahead with such toilsome descriptions of weights and pulleys? Against all appearances, the subtle arrangement of threads, weights, and pulleys of different forms, tables and rulers, was not aimed at avoiding the predictable failure of a material realization, but only for the purpose of safeguarding the possible exactness of the tracing. It is precisely by attempting to construct it that one would replace the terrain of exactness by the one of approximation, as it is done when one calculates the partial sum of an infinite series. Devoid of all effective utilization, the mechanism keeps its value*

[15] "I see that professor Bernoulli already has doubts about the geometricity of this generating process." (M.S. II p. 161)

[16] " I am happy that you acknowledge the scrupulous examination that I have applied to your general quadrature by motion. I have explained it to my brother, who might not have given it sufficient attention, was complaining of its obscurity. He now agrees with me and he warmly approved of your method." (M.S. III p. 138)

[17] M.S. II p. 164.

[18] "Also, I do believe that the description of the cycloid or of the lines generated by evolution is geometrical. And I do not see why one would restrict geometric lines to those whose equation is algebraic." (M.S. II p. 165)

*completely theoretical and temporary*[19]*, with respect to the stammering of geometry on the chapter of transcendentals. One more time, it is the grafting of the theoretical upon the mechanical which governs the Leibnizian postulation of a possible geometrical achievement, by means of several supplements and which conciliates the two antinomistic hopes of optimism.*

*Such an argument for resituating the mechanism of traction within the family of the developments and rotations that had brought to geometry such precious gifts as the cycloid, is valid for the construction of curves, but is it valid for the* problem *of quadratures? If the complex equipment described in the article has the status of a calculus and must accomplish its functions, what would be the usefulness of an unrealizable calculus? By leaving aside the possibility that Leibniz might have thought of a similar type of machine as his arithmetic machine, whose difficulties of realization were no less dissuasive, we must note, on the one hand, that the uncertainty between theory and application characterizing the notion of* construction *is equally enveloping that of* quadrature*, as in the case of an actual tracing by compass and straight edge, that of a calculus, or that of a simple operational skeleton of a calculus that one considers merely possible*[20]*. However, especially under the detailed technicality of the integraph, there lies an outline of a theoretical teaching of first importance: the rough sketch of a classification of the still ill explored problem of the inversion of tangents. In fact, let us observe that his first idea stems from the understanding of the relationship of* reciprocity *between derivation and integration, formalized by the differential calculus, but of which Leibniz here considers the geometrical translation. Within this perspective, no matter how detailed the equipment may be, it takes second rank behind the theoretical reformulation of the problem of quadratures that underlies it. The integraph* demonstrates *that making a quadrature comes down to tracing a curve whose tangents are already given. Its pedagogical interest is no less important. It has the advantage of* making visible *the tangents but also the infinitangular polygon. Moreover, the process of construction by traction that inspired him and that Claude Perrault's watch had given him the first utilization, puts into sight the envelop of a mobile line. Though he does not refer to his* De linea ex lineis*, it is quite probable that Leibniz had conceived the generality of this new mode of construction through the great kinship he saw between the different generations of envelopes, such as the caustics and evolutes. Therefore, as a pedagogical as much as a geometrical machine, the integraph acquires the pertinence of a materialized demonstration, at the moment when physics discovers the fertility of imaginary experiments.*

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[19] It is mostly this association, difficult to accept for us, which makes of the integraph a paradoxical machine.
[20] For example, the use of *quadrature of the hyperbola* as the set syntagma to designate logarithms.

The measurement of curves, surfaces, and most volumes, just like the determination of the centers of gravity, all comes down to quadratures of plane figures[21]; such is the starting point of {*geometry of measurements*}, which is by nature different from {*geometry of determination*}, which involves only lengths of straight lines and by their means determines unknown points from others that are given. As a rule, we may naturally reduce this geometry of determination to algebraic equations, whose unknown has a determined degree[22]. However, geometry of measurements is not, by nature, reducible to algebra, even if it happens, sometimes, that it is reduced to algebraic magnitudes (when we have to deal with ordinary quadratures); similarly, the geometry of determination is not of the domain of arithmetic, even when it happens (in the case where quantities are commensurable) that it is reduced to numbers, that is rational quantities. We can derive from this {*three types of magnitudes, rational, algebraic, and transcendental*}. The {*origin*} of algebraic {*irrationals*} resides in the {*ambiguity*}[23] of the problem, that is to say, its {*multiplicity*}; in fact, it would be impossible to regroup under one single calculus the different values or solutions of a problem, unless it is done by means of roots, but then, at the exception of certain particular cases, they cannot be reduced to rational magnitudes. On the other hand, the {*origin of transcendental magnitudes is infinity*}, so much so that the {*analysis*} which corresponds to the {*geometry of transcendentals*} (to which belongs measurements) is very precisely the {*science of the infinite*}[24].

Furthermore, when algebraic magnitudes are constructed, what is required are determined motions which do not need material curves but only straight edges, or, when material curves are considered, only their intersection[25] points are taken into consideration; on the other hand, in order to construct transcendental magnitudes, we have used, so far, the application[26] of curves with respect to straight lines, that is we have adjusted one to the other, as in the case of the construction of the cycloid, or with the unraveling of a thread, or a leaf, when they are wrapped around a curve or a surface. If

[21] Leibniz considered wrongly that a measure is that much simpler as the measured object is of a lesser dimension. He wrote to the marquis de l'Hospital: "I wish… I were able to reduce quadratures to the rectifications of curves. Because the dimension of the line is simpler than that of a space." (M.S. II p. 237)
[22] The confusion between these two radically different geometries and the two different types of analysis that it summons may be for Leibniz the reason for their illegitimate exclusions pronounced by Descartes.
[23] Which, just like that of ambiguous signs, it is not an indetermination but a proof of the fecundity of the characteristic.
[24] Leibniz defines here the program of his unfinished work *De Scientia Infiniti* (cf. P. Costabel, "*De Scientia infiniti*" in *Leibniz, aspects de l'homme et de l'oeuvre*, p. 105) and reveals why the urgent task of geometry concerns transcendentals : firstly the problems of integration, as a general rule, makes them arise, secondly, only the transcendental values truly manifest infinity: therefore, for example, as it was shown by the demonstration of the impossibility of an indefinite quadrature of a circular sector, the degree of the equation expressing this quadrature would exceed every finite degree. This character will be made explicit by the distinction between transcendental magnitudes and *interscendental* magnitudes (solutions of an equation of indeterminate degree but finite). To the extent that a transcendental equation were to be of an infinite degree, it would admit to an infinite number of solutions.
[25] For example, when one traces a circle in order to measure equal distances on a straight line.
[26] The term *applicatio* can be understood in two ways, on which Leibniz takes good care of maintaining the ambiguity: it could mean a material application by means of a thread (but, who has never used a thread in order to trace an evolute?) or it could mean a theoretical application, being reduced to a measure in the current sense of a rectification.

you wished to trace geometrically (that is by a constant and regulated motion) the Archimedean spiral, or the Quadratrix of the Ancients[27], you could do it without any difficulty by adjusting a straight line to a curve, in such a way that the rectilinear motion would be regulated from the circular motion. And that is why, contrary to what {*Descartes*} has done, I will not exclude such curves from geometry, because the lines which are so described are exact, and they involve properties which are very useful, and are adapted to transcendental magnitudes.

There exist, however, other means of constructing curves, which involve the addition of a physical component. Such would be the case when the solution of a problem of geometry of determination were to be found by means of light rays (which could be often done with great profit)[28], or if one were to proceed, as I have done with the quadrature of the Hyperbola, or for the construction of logarithms by making a uniform motion, and a retarded motion by a constant[29] rubbing motion, or else, by means of a

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[27] Also called the *Quadratrix of Dinostrate* or of *Hippias*. It is an auxiliary curve arising when the ancient geometers were looking for the quadrature of the circle. It is gotten by descending in a uniform speed BC while the radius AE rotates uniformly around A. Point F, the intersection of the Radius AE and line BC, trace the Quadratrix, which arrives at point G such that **arcBED/AB + AB/AG.**

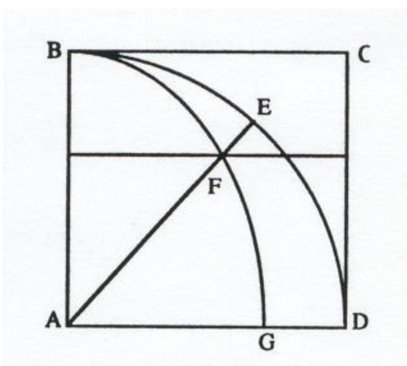


**Figure 28.**

Aside from the difficulty of adjusting these two motions, this construction runs mainly into the following objection: given that in the neighborhood of G, AE and line BC come to merge together, point G cannot be determined exactly. But, here, Leibniz only retains the possibility of describing a geometric curve by a mechanism which is theoretically exact, independently of its unrealizable character. Cf. G.J. Allman, Greek Geometry from Thales to Euclid, Dublin, 1889, p. 180-193; T. L. Heath, A History of Greek Mathematics, Oxford, 1921, t. I p. 225-230. As for the spiral of Archimedes, it can be gotten by the uniform motion of a point on a radius turning in a constant angular speed.

[28] Such an opening, no doubt inspired by the example of caustics, plunges the reader into a depth of perplexity. On can without a doubt think of a optical device that would suggest to the imagination the intuitive tracing of a curve, but how do you assign points or take measurements on a ray of light? Nevertheless, his remark tells us that in the mind of Leibniz, Mixed Mathematics are not reducible to the problems of kinematics and that the physical means of producing curves cannot be reduced to composition of motions as they have been known for a long time. One can think that the new curves constructed by means of optical devices are in particular the envelopes of families of curves, used in his construction of optical lines and of which he had initiated the theory in his *De linea ex lineis*.

[29] This construction is the subject of article I of *De resistentia Medii:* "if a mobile is animated by a motion that is composed of a uniform motion and of a retarded motion proportionately with (different) spaces, that is to say, if the mobile (M) is displaced on a rigid ruler (AB) in conformity with the present hypothesis

string, or a heavy chain which produce the Catenary, or the funicular curve (la chaînette). As long as the mode of its construction is exact, it will belong to theoretical geometry: as long as it is practical and useful, it has a right of application in practice[30]. Indeed, any motion executed according to determined hypothesis is as much of the domain of geometry as is a center of gravity.

But, there exist a new type of motion, which, I think, I have been the first to make use of in the constructions of geometric curves, and I will say in what circumstance; because it seems to me, better than any other type, to belong to pure geometry, and resembles the tracing of curves by means of threads originating from umbilical points or from focus curves[31]. In fact, the only condition that is required for the point to trace the curve in the plane is that it needs to be attached to the extremity of a thread located in the same plane (or in an equivalent plane), and that it must be in motion at the same time as the motion of the other extremity, but by a motion which is a simple traction, without any lateral impulse[32], which would not really work with a thread because of its flexibility; because the point has to be pulled in the direction of the tension of the thread which drags it along, that is in the direction of the thread to the extent that there exist no obstacle along the pathway.

However, since a material thread never has the absolute flexibility that geometry requires, it could easily drag an engraver's point, in other words, the point producing the trace (which is free in the plane), in such a way that the motion of the engraver's point would represent nothing else but a simple traction; but to this material obstacle we could easily oppose a material expedient[33] such that, when the tracing point is pressing down

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(which is being produced in a satisfactory manner by the rubbing of a ball moving horizontally along the ruler), while the extremity (B) of the ruler (AB), which is always parallel and has a uniform motion, is moving along the straight line (BT), this mobile will describe the logarithmic line (AL). Indeed, in general, if a mobile is moved by a motion which is composed of a uniform motion and a motion regulated by another law, it will describe a line whose ordinates and abscises express a relationship between the time and the spaces in that second law, which is a considerable theorem; we deduce from it a physical means of constructing logarithms, that ordinary geometry is incapable of constructing precisely." (M.S. VI p. 137) Here, what also makes the value of the process is not the precision that is to be expected of such a tracing, but the exactness of the idea of its generation.

[30] Here, Leibniz remodels the definitions of *De vera proportione Circuli*; he formerly called *mechanism* an approximate construction. His position has evolved on that point. A mechanism does not by itself pertain to practice. It can therefore be exact, in other words geometrical, as long as his description is. It becomes an approximation only when one tries to realize it in practice.

[31] This is an allusion to the tracing of tangents by the *method of nets* proposed by Tschirnhaus in his *Medicina Mentis*. The correction that Leibniz brings and which is published in the Journal des Savants of September 14, 1693, is inspired directly from the *Rule of composition of motions* published in the previous number: "We have to consider that the stiletto which extends the nets could be conceived as having as many directions with equal speeds between them, that there are nets: because he pulls them equally, and since he pulls them it is also pulled. Thus, its composed direction (which must be perpendicular to the curve) passes through the center of gravity in as many points as there are nets…And these points, because of the equality of the tendency in our case, are equally distant from the stiletto, and thus fall in the intersections of the circle and the nets." (M.S. VI p. 233 and P. Costabel, *Leibniz et la dynamique*, p. 108).

[32] This condition means that at any point, the thread is tangent to the curve and that consequently this curve will be considered as the envelope of the family of tangents.

[33] Cf. supra note 13. This phrase captures the whole ambiguity of the integraph.: does this mean that a material expedient would be able to reestablish the *absolute* exactitude of geometry?

slightly against the plane to which it belongs, it is bound to it; and, such an expedient can be represented by a weight added to the point, or tied to it, in such a way that, by this added heaviness, the point weighs on the horizontal plane where it is suppose to define the pathway which traces the curve.

In this way, if the resistance of the weight, which impedes the motion of the point, is always stronger than the small residue of stiffness which is subsisting in the thread, the thread would be that much more tense, and its motion that much more regular; thus, the added weight would help the point trace properly the curve by traction only, and with no lateral impulse, which is the only condition that had to be imposed for the required motion. Furthermore, it follows that such a motion is remarkably suited for transcendental geometry, because it directly involves tangents that indicate directions for the curves, in other words elementary magnitudes that are infinite in number, but with unassignable lengths, in other words infinitesimals.

It was a long time ago, in Paris, that I first imagined such a construction. The notorious Parisian Doctor, {*Claude Perrault*}, remarkable for his knowledge of Mechanics and Architecture, at the same time well known for his edition of Vitruvius, and who became one of the eminent life time members of the Royal Academy of Sciences[34], submitted to me, as well as to a lot of other people, the following problem, which he was not able to solve, as he honestly admitted: that is, find the curve BB which a heavy point traces in the horizontal plane, at point B, or some equivalent point, and which is attached at the extremity B of a thread, or of a small chain AB; and when, by guiding the other extremity A of the thread AB along a fixed straight line AA, the weight of B is being pulled in the horizontal plane[35], (or in another equivalent plane) where the straight line AA is already located along with the motion of the thread AB.

In order to make the causality of this process more intelligible, he was using a watch B in a silver jewel-case that he had attached to a small chain whose other extremity he would pull along a straight edge AA that was fixed on a table. In this manner, the lowest part of the jewel-case (located in the middle of the bottom part) was describing on the table the curve BB. When I closely examined that curve, (It was in the period when I was studying tangents), I immediately realized that the key to solving the problem resided in the fact that the thread was constantly tangent to the curve[36], that is to say that any straight line $_{3}A_{3}B$ is tangent to curve BB at Point $_{3}B$.

[34] This doctor, Claude Perrault, is the brother of the famous writer. His father intended him to be in the medical profession, a carrier from which he rapidly turned away in order to study architecture. In 1666, he became one of the founding members of the Academy of Sciences. Colbert put him in charge of translating Vitruve of whom we only had incomplete commentaries. This was a delicate work that he accomplished with a lot of talent, and for which he became famous. His main works in architecture ( the Louvre colonnade and the observatory of Paris) are no less remarkable.

[35] The Gerhardt edition has here directum instead of dictum.

[36] The tone of the confidence makes us think that the watch of Perrault could have played a significant role in the generation of these most profound intuitions of the Leibnizian differential calculus. On the one hand, it leaves on the table the trace of an envelope of a mobile line; on the other hand, it visualizes the *inversion* of the problem of the tangents. Ordinarily, it is the curve which is visible and the tangents that are to be sought. Here, it is the successive positions of the chain that show the tangents, and it is the curve that must be sought for.

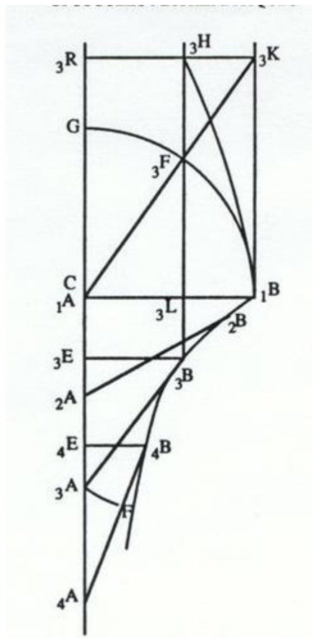


Figure 29.

Here is a demonstration of this: draw an arbitrarily small circular arc ${}_3AF$, whose center is ${}_3B$, and whose radius is the thread ${}_3A{}_3B$. Pull the thread ${}_3BF$ from F, directly, that is to say, according to its own direction up to ${}_4A$, in such a way that from ${}_3BF$ the thread goes to ${}_4B{}_4A$; assuming that we proceeded for points ${}_1B$ and ${}_2B$ as we did for ${}_3B$, any point B would describe a polygon ${}_1B{}_2B{}_3B$ etc., whose sides always fall on the thread; thus, by indefinitely diminishing arc $A{}_3F$, and by finally making it vanish, we describe our tracing by the motion of continuous traction, where the lateral displacement of the thread is continuous but always {*unassignable*}, it is clear that the polygon is changed into a curve, which has this thread as its tangent[37]. So, I realized by this, that the question could be reduced to a problem of conversion of tangents: find a curve BB such that the portion of its tangent between the axis AA and the curve BB is equal to a given constant[38]. It was not difficult either to understand that the tracing of this curve could also be reduced to the quadrature of the hyperbola.

Let us draw, in fact, the circle ${}_1BFG$ whose center is C, or A (where the thread $A{}_1B$ is at the same time the ordinate of the curve and its tangent), and whose radius is

[37] One can understand the attraction for Leibniz of this apparatus which visualizes his *great principle* of the geometry of measurement, the equivalence between a curve and an *infinitangular* polygon (cf. *De dimensionibus figurarum inveniendis*) So, to the extent that H.J.M. Bos emphasizes its connection with the characteristic property of the Leibnizian differentials, that is to say, their indetermination (which is translated here by the possibility of taking the curvilinear elements of abscises for the variable of reference), the Leibnizian construction suggests a link between this indetermination and the idea of an envelope of a family of curve.

[38] Such is the formulation of the problem as treated by Huygens in the *Journal de Rotterdam.*

AB. This circle cuts the axis AE; given that $_1$BK is parallel to this axis, which the straight line CF cuts at K, $_1$BK will be tangent to the circular arc $_1$BF. Then, trace the straight line FLB going through F, and parallel to the axis AE, that line cuts $_1$A$_1$B at L, and curve BB in B, from this straight line, trace LH equal to $_1$BK; by proceeding everywhere in the same manner, we get the curve of the tangent $_1$BHH, and we realize that rectangle $_1$B$_1$AE is equal to the figure of the tangents, that is to the trilateral area $_1$BLH$_1$B[39]; for example the product of $_1$B$_1$A and $_1$A$_3$E will be equal to the triline $_1$B$_3$L$_3$H$_1$B. Therefore, since we can find the area for the figure of the tangents by the quadrature of the hyperbola, that is to say, by logarithms, as everybody knows, it is also clear that we can equally get $_1$A$_3$E and $_3$L$_3$b, and consequently, any point $_3$B on the curve. Conversely, if we are given the curve BB, we will be able to construct the quadrature of the hyperbola, that is, the logarithms.

I don't want to explain all of this more extensively than what is necessary primarily because, in my opinion, the well known {*Christian Huygens*} has perfectly covered the subject; he told me, in a recent letter, that he just got an original idea for the quadrature of the hyperbola[40], which has been recently published in the History of the

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[39] According to the characteristic property of the curve, $_3$A$_3$B =$_1$A$_1$B, then $_1$A$_3$F =$_1$A$_1$B therefore $_1$A$_3$F = $_3$A$_3$B, the straight lines $_1$A$_1$F and $_3$A$_3$B are parallels. Given $_1$BL the abscissa x of curve BB, and L$_3$B its ordinate y. Given δ the angle of the tangent with the axis of the abscissas: tg δ = dx/dy; therefore here tgδ= $_1$BK/$_1$A$_1$B. Therefore the curve HH represents for each x the value of dx/dy (in modern language, by posing that y = f(x) , HH is the curve of the derivative f(x) ) On this figure of tangents , that Leibniz also calls *figure of canonical tangents of the circle*, (M.S. II p. 164) , the area $_1$BLH$_1$B is measured by a primitive of f(x), or either by f(x). The product of AB by $_1$AE (=LB) is therefore equal to the area $_1$BLH$_1$B. On the other hand, one can also note that this area is calculable by logarithms, even though the results are less immediate than Leibniz suggests. In fact, by calculating the coordinates of curve HH, thsnks to angle δ, one obtains : x = AB(1-cosδ), and y = Abtgδ. The area 1BLH1B would then be calculated by the primitive. :

$$\int ydx = AB^2 f tg(\delta).\sin(\delta).d\delta = AB^2 \left[ \int d\delta/\cos\delta - \int \cos\delta.d\delta \right]$$

The second term fcosδ.dδ is simply expressed as sinδ; as for the first term, let us note that it corresponds to the *sum of the secants*, that in the loxodormic curve, Leibniz has calculated in function of logarithms. It is therefore possible that Leibniz is making reference to that result. In modern terms, $\int d\delta/\cos\delta =$ log 1tg(δ/2 + x/4) 1. The parametric equation of the curve is therefore x = k(1-cosδ);y = k[log[tg(δ/2 + x/4)] –sinδ] for δ varying from 0 to x/2.

[40] "I will tell you some other time about the physico-mathematical quadrature of the hyperbola that I have found not long ago, the speculation on which has something pleasant." (M.S. II p.153) The following letter testifies to the importance that Huygens attached to it: "I would like to have your appreciation of my Tractoria for the quadrature of the hyperbola…about which there is this remarkable thing that following the rules of mechanics, assuming that the plane is horizontal, the description must be perfect and consequently the same with the quadrature and its mean (of generation)." (M.S. II p. 161) That construction is published in the *Journal de Rotterdam* that Leibniz was not able to get. His answer contains precious indications about his conception of geometricity: "The construction of lines that you call Tractoria is of importance. I would rather use that term for the construction rather than for the line, because all lines can be constructed in that manner, always taking in the tangent a point whose distance from the point of the curve may be given, which will produce a new line, along which a piece of thread being attached, the other (point) will describe the given curve…And since you have meditated on the means of making (that motion) exact and practical, you will find that there might not be a better (name) in geometry which is better suited…When someone asks if that construction is geometrical, , it is necessary to agree on its definition. According to my language, I would say that it is. Thus, I believe that the description of the cycloid or of your lines

Works of Scientists. I can get a good idea of it thanks to the articles that the {*Bernoulli*} brothers have just published in the {*Acta Eruditorum*}, because, starting from the discoveries of Huygens, they have very judiciously applied a similar motion to describe the curve for which the portion of the tangent between the curve and the axis, that is BD, is in constant relationship with a portion of the axis between a fixed point and the point of intersection of the tangent, that is AD[41], as is the constant proportion between two straight lines, N and M. This is what convinced me to finally publish my old reflections on this subject.

constructed by evolution is geometrical. And I don't see why one would restrict geometric lines to those who equation is algebraic." (M.S. II p. 164)

[41] This problem has been proposed by Jean Bernoulli in an article of May 1693 entitled *Solutio problematis Cartesio quondam propositi a DN. De Beaune:* "Find curve ABC which has the following property: once a tangent is traced from a given point to the axis AE, let the abscissa AD be with tangent BD in a constant proportion with M and N." (figure 30) with the following commentary: "Whatever the ratio between M and N, one can always describe a curve ABC by a continuous motion with the same easiness, independent of the fact that, according to the ratio of M and N, the curve might become more and more complex; indeed, in the case of equality, one can readily see that the curve ABC would be a circle." (*Opera Johannis Bernoulli*, t. I p. 65) The formulation retained by Leibniz is more general, because the fixed point is no longer necessarily identified with the origins of abscissas. Jacques Bernoulli immediately published the solution and proposes a practical construction by means of "the constant motion of a thread GCB including the point at extremity C…" Huygens had begun to steer away from the problem but, as he admitted to Leibniz, he was unable to avoid the fact that "it kept rolling in his head.", and in October 1693, he gave a ciphered solution in the Acta: "As far as the tracing of the curves in question onto a plane are concerned, I could, if it were necessary, indicate tracings that are different from the one proposed by the illustrious Bernoulli, and maybe more practical, and equally show how my tracing of the quadrature of the hyperbola has been perfectly realized,  which should be considered as the simplest of all Tractorias (this the name that can be given to them) , to the extent that its generation does not require the use of threads, but of only a rod equipped with a point fixed on each side, in a manner such that we can proceed in reverse by checking to see if the tracing has been done correctly. But I think that we have to wait for the elucidation of a remarkable application of those lines." (M.S. V p. 290)

Leibniz had himself published the following commentary: "I have very much appreciated the problem posed by Bernoulli last May…especially because those very people who have not mastered the rudiments of my differential method will not succeed right away. Not only can on succeed bu a motion but also by an analytical calculus, if the ratio between the product of two straight lines (the tangent t and the portion of the axis r), or of their powers and of a power of the chord AB which underlies them, homogeneous to this product, , for example, the report between tr and $c^2$, or between trr and $c^3$, or some other. The same problem applies to a lot of other relations, for example, if we knew the ratio between the portion AD and the ordinate BE." (*Ad problema in Actis Eruditorum an. 1693 mense majo propositum*, M.S. V p.288)

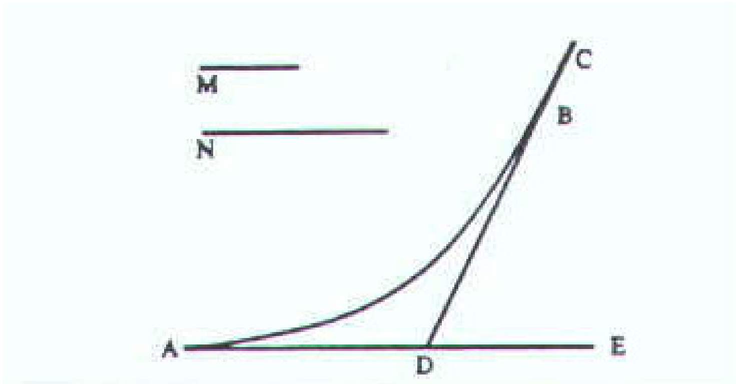


figure 30.

Right away, it was easy for me to understand that, once you grasped the relationship between the motion and the tangents, then you could use the same process to construct many other curves which would otherwise be more difficult to square. Because, even if you were to suppose that AA is not a straight line but a curve, the thread would nonetheless be tangent to BB. Furthermore, even if the length of the thread AB were to increase or diminish while you are pulling on it, it would nonetheless remain tangent to it. That is why, whatever the relationship between CA and AB (for example if the AB's were the sinuses and the CA's were the corresponding tangents), several expedients could be used to regulate the motion of the thread so that it could move, while itself diminishing, according to a given law.

This procedure gives us the ability to trace an infinite number of solution curves representing a similar problem, for example, the curve that goes through a given point. If the point producing the tracing is being pulled simultaneously by several threads, you will generate several composite directions[42]. But, even if we were dealing with only one thread, we could vary its length by attaching to weight B, a wheel, or a rotating mechanism, such that it could describe a cycloid in the plane. We could also join to B a rigid straight edge which would be always perpendicular to the thread, or which would make a constant angle with it, or an angle which would vary according to some determined law, and consider finally the tracing produced by another mobile point on that straight edge.

We can also pull two weights simultaneously from the same plane, whether their distances are fixed, or variable, during the motion in the plane. We can also imagine two

[42] In September 1693, Leibniz published in *Acta Eruditorum* his *General Rule for the Composition of Motions* (M.S. VI p. 231) One of the problems consists in finding the tangent to a curve described by several extended nets: "This tangent corresponds to the composed direction of the stiletto." In order to discover it, Leibniz proposed the description of a circle around the point of the curve, cutting the nets at points B, C, D…" If you find the center of gravity of these points, that is G, a straight line drawn from A, the normal AG will be the sought for tangent." Even though it remains very obscure, Leibniz proposes to extend this method to ordinary conics, to the ovals of Descartes, to the co-evolutions of Tschirnhaus, and to an infinite number of other lines. Cf. P. Costabel, Leibniz et la dynamique, 1692, p. 56-95.

planes, one in which point C[43] is invariably linked to one plane, and the other where the engraver's point B is tracing with a very light contact (which does not interfere with the motion of B) and describes a new curve, and then suppose that this second plane has its own motion; the tangents to this new curve would be nothing else but the straight line which gives direction to the composite motion of the engraver's point in the fixed plane, and the motion of the other plane. We shall determine the properties of the tangents from this new curve so described.

By meditating on the extreme generality of this type of motion and on the incalculable applications that it can offer, I have darkened many a paper, years ago, while meditating on their practical applications, with respect to the wonderful resources that I was noticing with the conversion of tangents, and even more within the quadratures. Having discovered a {*construction*}, which spontaneously extends, in a totally universal manner, to all of the quadratures, I don't even know if, on this question, the latest developments in Geometry have established a more general one. I have resolved myself to publish it. Since up until now I had reserved it, as raw material, for a future work, and with the idea of developing an integral theory[44], my other tasks of totally different interests have taken me so much that I must, at the first opportunity, discharge myself of these old ideas for fear of loosing them; these current results, which I have waited twice as long as the time recommended by Horace, I have waited for {*Lucine*} long enough[45].

I would now like to show that {*the general problem of quadratures is determined by constructing a curve whose inclination obeys a given law*[46]}, that is to say, against which the assignable sides of the characteristic triangle have between them a given relationship; after which I will show that {*we can construct this curve by the motion that I have imagined*}. In fact, for any curve C(C), I imagine {*two self-similar characteristic triangles*}, one assignable TBC, the other, unassignable GLC.

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[43] The Acta reported CB instead of C, an error corrected by Gerhardt.

[44] Cf. P. Costabel, "De scientia infiniti", in *Leibniz, aspects de l'homme et de l'œuvre*, p. 105.

[45] Lucine, the goddess presiding over childbirth, is sometimes identified with Juno sometimes to Diana. In his *Poetical Art*, Horace recommends that poets should wait a period of nine years before publishing their works, which bring the discovery of Leibniz back to 1674-75.

[46] Here, Leibniz established explicitly the relation of reciprocity between the calculus of quadratures and that of the tangents (in modern terms, between integration and derivation) What follows shows that he is mostly thinking of the relation between the problem of inversion of tangents and the calculus of quadratures, but a letter to l'Hospital shows us that he was thinking of constructing the *inverse of tangents* "independently of quadratures" by a method pertaining to envelopes, since Leibniz proposes to reduce the inversion of tangents to the tracing of a curve which is perpendicular to a series of given lines (M.S. II p. 238). The merit of this intuition is attested by the fact that, in a modern analysis, an envelope of integral curves of a differential equation of the first degree is equally an integral curve of that equation and can therefore provide it with a singular solution.

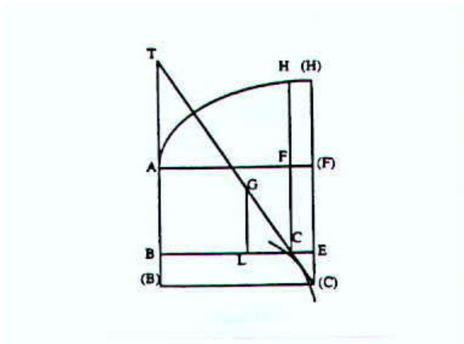


figure 31

The {*unassignable*} triangle is delimited by the elements GL and LC of the coordinates[47] CF and CB which form its sides, and by the elementary arc GC, which constitutes its base or its hypotenuse. But the assignable triangle TBC is delimited by the axis, the ordinate and the tangent, and expresses consequently the angle that the direction of the curve makes (that is also the direction of its tangent) with the axis or its basis, in other words, the inclination of the curve at the given point C.

The task, here, is to square the surface F(H) located between the curve H(H), the two parallel straight lines FH and (F)(H), and the axis F(F); take on this axis a fixed point A, and take an a conjugated axis the straight line AB perpendicular to AF, and going through A, then take on each straight line HF (which can be extended at will) a point C, that is let's construct the curve C(C) in such a way that, once you have drawn between C and the conjugated axis AB (extended if need be), the conjugated ordinate CB (equal to AF), but also the tangent CT, the portion of the axis TB, that they delimit should be with BC in the proportion of HF with the constant a, that is to say, such that the product of a by BT is equal to the rectangle AFH (which circumscribes the triline AFHA).

Under such conditions, I say that the product of a by E(C) (the difference between the ordinates FC and (F)(C) of the curve) is equal to the surface F(H); from this fact, if we extend the curve H(H) all the way to A, the triline AFHA of the figure to be squared is equal to the product of the constant a by the ordinate FC of the squaring figure. My calculus shows this immediately. Suppose in fact that AF = y, FH = z, BT = t, FC = x, then you get by hypothesis t = zy/a, but also t = ydx/dy, the expression of the tangents according to the formulas of my calculus, so: adx = zdy and consequently ax = Function of zdy = AFHA. The curve C(C) is therefore a quadratrix with respect to the curve H(H), because the product of its ordinate FC by the constant a is equal to the area under the curve, that is to the summation of the ordinates H(H) applied to their respective abscissa AF[48].

Furthermore, since BT is to AF in the same proportion as FH is to a (hypothetically), and since the relationship of FH to AF (relationship which requires the

[47] The term has been introduced in *De linea ex lineis.*
[48] Here, Leibniz resuscitated the ancient general rule of the geometry of indivisibles (cf. supra *De geometria recondita*, note 40).

figure to be squared) is given, we shall also know the relationship of BT to FH, that is to say, the relationship of BT to BC, and therefore, the one of BT to TC, in other words, the relationship between the sides of triangle TBC. That is why, the task of determining all of the quadratures and all of the measurements are realized, once you are able to establish the relationship of the sides of the assignable characteristic triangle TBC, in other words, from the law of the inclines, when you trace curve C(C), since we have shown that it is a quadratrix.

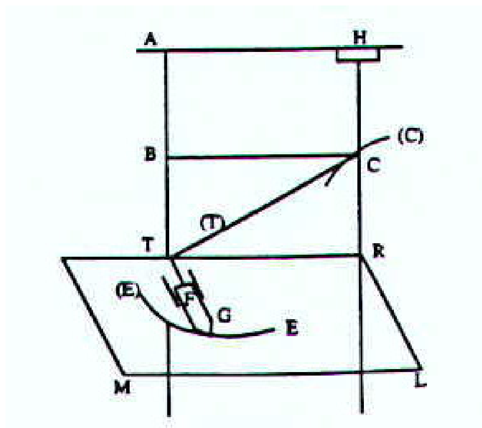


figure 32.

The tracing can be generated in the following manner: construct figure 32, by establishing TAH as a fixed right angle located in the horizontal plane; have an empty cylinder TG move along the side AT vertically under the stated horizontal plane. Inside of this empty cylinder, place a second filled cylinder FE, mobile from top to bottom, and which is connected at the top of F to a string FTC in such a manner that the part FT is located inside of the empty cylinder while the part TC is located in our horizontal plane. Furthermore, point C which describes the curve C(C), at the end of the string TC, must be bound to the plane by means of a weight which will rest on it; but the beginning of the movement will be located in the empty cylinder TG which, while coming away from A along AT will attract C. And while the point traces the curve, that is to say the engraver's point C will move ahead toward A, a ruler HR located in the horizontal plane which is perpendicular to AH (the other side of the fixed right angle TAH), this impulsion would not impair the motion of point C, which is moving solely by the traction of the string, and would therefore allow it to maintain the direction of its motion. Let us suppose, as well, the existence of a shelf RLM which is moving downward at right angle with respect to the ruler HR, at the common point R, and which is otherwise constantly pushed by the empty cylinder, in such a way that ATHR forms a rectangle. Let us suppose finally that on this shelf is traced (on a thin relief sheet) a rigid line E(E) on which the filled cylinder bites continuously through a notch which you can imagine is being engraved in the extremity E, such that as R approaches T, the cylinder FE descends. Since the length ET + TC is given (It includes the filled cylinder EF and the totality of the thread TC), and also the relationship between TC and TR or BC (because the law of the inclines is given), we will also know the relationship between ET and TR, the ordinate and the abscissas of line E(E), then we can determine, by means of ordinary geometry alone, and trace it on the shelf LRM.

Thus, this device gives us the tracing of the curve C(C). So, by the very definition of our motion, TC is everywhere tangent to curve C(C). Here, then, is the construction of a curve C(C) whose inclinations obey a given law, that is to say where is given the relationship between the sides of the assignable characteristic triangle TRC or TBC.  This curve being a quadratrix of the figure that was to be squared, as I have just shown it, we will also obtain the quadrature by the same required measure.

We can apply similar procedures in varied ways to different problems posed by the method of inversion of tangents: for example, if point T were to be displaced on a curve T(T) (as opposed to straight line AT), the coordinate HC would have intervened in the calculation as well (that is to say the abscissas AB). In point of fact, the whole problem of inversion of tangents can be reduced to a relationship between three lengths, that is, the two coordinates CB and CH and the tangent CT, or other functions[49] instead.

However, we can often end the discussion by a motion which is much more simple. For example, if the relationship between AT and TC had been known, (that is to say, given a family of circles, given in ordered position, and cutting a curve at right angle, find that curve), a much simplified apparatus would have sufficed. Since nothing concerning H and R are to be considered, all you need to do is to trace a rigid directrix line E(E) in a fixed vertical plane passing through AT. Thus a point T[50] being chosen on the fixed straight line AT, that is to say, once the empty cylinder TG is determined, when the filled cylinder TE comes down as required by the directrix line E(E) on which it is biting, because the summation of ET + TC is constant [51](as before) and that the relationship between AT and TC is given, we will easily discover the adequate relationship between AT and TE, that is to say, the nature of line E(E) that permits you to find the curve C(C) which we were looking for.

[49] Cf. infra *Nova calculi differentialis applicatio*. Leibniz identifies as a *function* any straight line segment that can be determined from a curve.
[50] The text of the *Acta* has TC instead of T.
[51] The text of Gerhardt has ET + TF instead of ET + TC, also further on, AT and TC instead of AT and TE.